A STRONG MAXIMUM PRINCIPLE FOR WEAK SOLUTIONS OF QUASI-LINEAR ELLIPTIC EQUATIONS WITH APPLICATIONS TO LORENTZIAN AND RIEMANNIAN GEOMETRY

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Abstract. The strong maximum principle is proved to hold for weak (in the sense of support functions) sub- and super-solutions to a class of quasi-linear elliptic equations that includes the mean curvature equation for C^0 spacelike hypersurfaces in a Lorentzian manifold. As one application a Lorentzian warped product splitting theorem is given.

Contents

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1. Introduction

The geometric maximum principle for C^2 hypersurfaces, first proven by Alexandrov [2, 3], has become a basic tool in differential geometry especially for proving uniqueness and rigidity results. The geometric maximum principle is a consequence of the strong maximum principle of Alexandrov [1] for C^2 sub- and super-solutions for nonlinear uniformly elliptic equations which is in turn based on the original maximum principle of E. Hopf [20] for linear elliptic equations.

However there are naturally occurring geometric situations, for example level sets of Busemann functions, where the classical maximum principle does not apply as the hypersurfaces involved are only of class C^0 (that is locally graphs of continuous functions) and only satisfy curvature inequalities in the sense of support functions (see Definitions 2.3, 3.3, and 3.9). In this paper we give a version of the strong maximum principle, Theorem 2.4, and deduce geometric maximum principles, Theorems 3.6 and 3.10, general enough to cover most recent applications of maximum principles to uniqueness and rigidity questions in Riemannian and Lorentzian geometry.

Our main analytic result is Theorem 2.4 which extends the strong maximum principle to weak (in the sense of support functions) sub- and super-solutions of uniformly elliptic quasi-linear equations. For linear equations this is due to Calabi [9] who also introduced the idea of sub-solutions in the sense of support functions. The geometric versions are a strong maximum principle, Theorem 3.6, for C^0 spacelike hypersurfaces (Definition 3.1) in Lorentzian manifolds and Theorem 3.10 which is a maximum principle for hypersurfaces in Riemannian manifolds that can be locally represented as graphs. Eschenburg [12] gives a version of these geometric results under the extra assumption that one of the two hypersurfaces is C^2, but in the applications given here neither hypersurface will have any *a priori* smoothness.

A very natural example of rough hypersurfaces where our maximum principles work well is the level sets of Busemann functions in Riemannian or Lorentzian manifolds. By applying our results to these level sets in Lorentzian manifolds we prove a warped product splitting theorem which can be viewed as an extension of the Lorentzian splitting theorem [11, 15, 23] to spacetimes that satisfy the strong energy condition with a positive cosmological constant. (See §4 for a discussion of how proofs of splitting theorems and warped product splitting theorems are simplified by use of the maximum principle for rough hypersurfaces.) As applications of warped product splitting we obtain a characterization, Corollary 4.4, of the universal anti-de Sitter space which can be viewed as a Lorentzian analogue of the Cheng maximum diameter theorem [10]. Certain Lorentzian warped products that are locally spatially isotropic are characterized (Theorem 4.6) by the existence of a line together with a version of the Weyl curvature hypothesis of R. Penrose. Further applications of maximum principles for rough hypersurfaces are given in [5, 14, 21].

2. The Analytic Maximum Principle

We denote partial derivatives on $\mathbf{R}^n$ (with coordinates $x^1, \dots, x^n$) by

$$D_i f := \frac{\partial f}{\partial x^i}, \qquad D_{ij} f := \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

Also let

$$Df := (D_1 f, \dots, D_n f), \qquad D^2 f := [D_{ij} f].$$

Thus Df is the gradient of f and $D^2 f$ is the matrix of second order partial derivatives of f, i.e. the Hessian of f. For each point x in the domain of f the Hessian $D^2 f(x)$ is a symmetric matrix. If A and B are symmetric matrices we write $A \le B$ if $B - A$ is positive semi-definite.

As coordinates on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ use $(x, r, p) = (x^1, \dots, x^n, r, p^1, \dots, p^n)$. If $U \subseteq \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ the **fiber** over $x \in \mathbf{R}^n$ is

$$U_x := \{(r, p) \in \mathbf{R} \times \mathbf{R}^n : (x, r, p) \in U\}.$$

Note that the fiber may be empty.

Definition 2.1. Let Ω be an open set and $U \subset \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$. Then a C^1 function $u : \Omega \to \mathbf{R}$ is **U–admissible** over Ω if and only if for all $x \in \Omega$, $(x, u(x), Du(x)) = (x, u(x), D_1 u(x), \dots, D_n u(x)) \in U$. (In particular this implies the fiber $U_x \ne \emptyset$ for all $x \in \Omega$.) $\square$

Let $U \subseteq \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ be open. Then a **quasi-linear operator** $\mathcal{M}$ on U is a collection of functions $a^{ij} = a^{ji}, b : U \to \mathbf{R}$ $1 \le i, j \le n$. If u is a U–admissible C^2 function defined on the open set $\Omega \subseteq \mathbf{R}^n$, then $\mathcal{M}$ is evaluated on u by

$$\mathcal{M}[u] = \sum_{i,j} a^{ij}(x, u, Du) D_{ij} u + b(x, u, Du).$$

Definition 2.2. The quasi-linear operator $\mathcal{M}$ on the open set $U \subseteq \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ is **uniformly elliptic** on U if and only if

1. The functions a^{ij} and b are continuous on U and are C^1 functions of the arguments (r, p).
2. There is a constant $C_{\mathbf{E}}$ so that for all $(x, r, p) \in U$

$$\frac{1}{C_{\mathbf{E}}} \|\xi\|^2 \le \sum_{i,j} a^{ij}(x, r, p) \xi_i \xi_j \le C_{\mathbf{E}} \|\xi\|^2 \tag{2.1}$$

and

$$\left| \frac{\partial a^{ij}}{\partial p^k} \right|, \ \left| \frac{\partial a^{ij}}{\partial r} \right|, \ \left| \frac{\partial b}{\partial p^k} \right|, \ \left| \frac{\partial b}{\partial r} \right|, \ |b| \le C_{\mathbf{E}}. \tag{2.2}$$

The constant $C_{\mathbf{E}}$ is the **constant of ellipticity** of $\mathcal{M}$ on U.

$\square$

We will need the notion of a support function. Let u be defined on the open set $\Omega \subset \mathbf{R}^n$ and let $x_0 \in \Omega$. Then φ is an **upper support function** for u at x_0 if and only if $\varphi(x_0) = u(x_0)$ and for some open neighborhood N of x_0 the inequality $\varphi \geq u$ holds. The function φ is a **lower support function** for u at x_0 if and only if $\varphi(x_0) = u(x_0)$ and $\varphi \leq u$ in some open neighborhood N of x_0. Given a sequence $B^1, \dots, B^n$ of functions set

$$\|B\|(x) := \sqrt{(B^1(x))^2 + \cdots + (B^n(x))^2}$$

and if φ is a C^2 function we set

$$|D^2\varphi|(x) := \max_{i,j} |D_{ij}\varphi(x)|.$$

Definition 2.3. Let $U \subseteq \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ be open, let $\mathcal{M}$ be a uniformly elliptic operator on U and let H_0 be a constant.

1. A lower semi-continuous function u defined on the open subset $\Omega \subseteq \mathbf{R}^n$ satisfies $\mathcal{M}[u] \leq H_0$ **in the sense of support functions** iff for all $\varepsilon > 0$ and $x \in \Omega$ there is a U–admissible C^2 upper support function $\varphi_{x,\varepsilon}$ to u at x so that
$$\mathcal{M}[\varphi_{x,\varepsilon}](x) \leq H_0 + \varepsilon.$$
2. An upper semi-continuous function v defined on Ω satisfies $\mathcal{M}[v] \geq H_0$ in the **sense of support functions with a one–sided Hessian bound** iff there is a constant $C_{\mathbf{S}} > 0$ (the support constant) so that for all $\varepsilon > 0$ and $x \in \Omega$ there is a U–admissible C^2 lower support function $\psi_{x,\varepsilon}$ for v at x so that
$$\mathcal{M}[\psi_{x,\varepsilon}](x) \geq H_0 - \varepsilon, \quad \text{and} \quad D^2\psi_{x,\varepsilon}(x) \geq -C_{\mathbf{S}} I$$
(The constant $C_{\mathbf{S}}$ can depend on v but is independent of ε and x.)

□

We are now ready to state the main theorem of this section.

Theorem 2.4 (Maximum Principle)**.** *Let $\Omega \subseteq \mathbf{R}^n$ be open and connected and let $U \subseteq \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ be such that for each $x \in \Omega$, the fiber U_x is nonempty and convex. Let $u_0, u_1 : \Omega \to \mathbf{R}$ be functions satisfying the following.*

1. *u_0 is lower semi-continuous and $\mathcal{M}[u_0] \leq H_0$ in the sense of support functions.*
2. *u_1 is upper-semi-continuous and $\mathcal{M}[u_1] \geq H_0$ in the sense of support functions with a one–sided Hessian bound.*
3. *$u_1 \leq u_0$ in Ω, and there is at least one point $x \in \Omega$ with $u_1(x) = u_0(x)$.*

Then $u_0 \equiv u_1$ in Ω and $u_0 = u_1$ is locally a $C^{1,1}$ function in Ω. Finally if a^{ij} and b are locally $C^{k,\alpha}$ functions in U then $u_0 = u_1$ is locally a $C^{k+2,\alpha}$ function in Ω. In particular if a^{ij} and b are smooth, then so is $u_0 = u_1$.

Remark 2.5. The geometric maximum principle for C^2 sub- and super-solutions to the Laplacian $\Delta := \sum_i D_{ii}$ is classical and follows from the fact that the difference $f = u_1 - u_0$ is sub-harmonic and use of the sub-mean value theorem for sub-harmonic functions. For C^2 sub- and super-solutions to general variable coefficient linear operators the result is due to E. Hopf [20]. The extension to nonlinear uniformly elliptic equations was done by Alexandrov [1, 3]. The formulation and proof of the strong maximum principle for linear operators with sub- and super-solutions defined in the sense of support functions is due to Calabi [9]. A partial geometric extension of Calabi's result to the mean curvature operator of hypersurfaces in Riemannian and Lorentzian manifolds was given by Eschenburg [12] who also introduced (in a geometric context) the idea of $\mathcal{M}[u_0] \geq H_0$ in the sense of support functions with a one sided Hessian bound. □

The proof of Theorem 2.4 will be given in §2.4. An outline of the proof is given in §2.1 and some preliminaries are given in §2.2 and §2.3.

2.1. **Outline of the Proof.** Here we give the ideas behind the proof which, stripped of the technical details, are easy. The basic outline of the proof and the idea of using support functions is as in Calabi [9] which in turn owes much to the original paper of E. Hopf [20].

First let $\mathcal{M}$ be as in Theorem 2.4 and let φ_0 and φ_1 be C^2 functions. Then (Lemma 2.12 below) there are functions $A^{ij}[\varphi_0, \varphi_1]$, $B^i[\varphi_0, \varphi_1]$, $C[\varphi_0, \varphi_1]$ so that

$$\begin{aligned}\mathcal{M}[\varphi_1] - \mathcal{M}[\varphi_0] = \sum_{i,j} A^{ij}[\varphi_0, \varphi_1] D_{ij}(\varphi_1 - \varphi_0) + \sum_i B^i[\varphi_0, \varphi_1] D_i(\varphi_1 - \varphi_0) \\ + C[\varphi_0, \varphi_1](\varphi_1 - \varphi_0)\end{aligned} \tag{2.3}$$

and such that the matrix $\left[A^{ij}[\varphi_0, \varphi_1]\right]$ is pointwise positive definite. Given φ_0 and φ_1 define the linear differential operator L_{φ_0,φ_1}

$$L_{\varphi_0,\varphi_1} := \sum_{i,j} A^{ij}[\varphi_0, \varphi_1] D_{ij} + \sum_i B^i[\varphi_0, \varphi_1] D_i. \tag{2.4}$$

With this notation and the hypotheses of Theorem 2.4 assume, toward a contradiction, that $u_0 \neq u_1$. Then $K := \{x \in \Omega : u_1(x) = u_0(x)\}$ is a proper closed subset of Ω. Thus there is a closed ball $\overline{B}(x_0, 2r_0)$ so that $\overline{B}(x_0, 3r_0) \subset \Omega$ and $\overline{B}(x_0, 2r_0)$ meets K in exactly one point x_1 and this point is on $\partial B(x_0, 2r_0)$. Let $r_1 \leq r_0$ and let

$$S' := \partial B(x_1, r_1) \cap \overline{B}(x_0, 2r_0), \quad S'' := \partial B(x_1, r_1) \setminus \overline{B}(x_0, 2r_0),$$

see Figure 1. Then $(u_1 - u_0) < 0$ on S' as S' is disjoint from K. For $\alpha > 0$ set $w(x) := \|x - x_0\|^{-\alpha} - \|x_1 - x_0\|^{-\alpha} = \|x - x_0\|^{-\alpha} - (2r_0)^{-\alpha}$. As w is bounded on the compact set S' and $(u_1 - u_0)$ is negative on S' it is possible to choose $\delta > 0$ so that if h is the function $h := (u_1 - u_0) + \delta w$ then $h < 0$ on S'. But w is a decreasing function of $\|x - x_0\|$ (so $w(x) < 0$ on S'') and $(u_1 - u_0) \leq 0$ so also $h < 0$ on S''. Thus $h < 0$ on $S' \cup S'' = \partial B(x_1, r_1)$ and

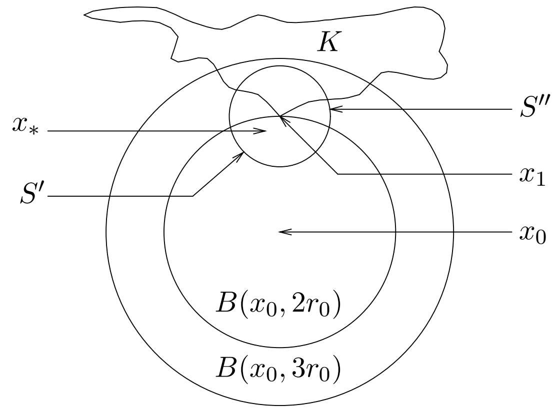

FIGURE 1.

$h(x_1) = (u_1(x_1) - u_0(x_0)) + \delta w(x_1) = 0$, so the function h has a nonnegative interior maximum in $\overline{B}(x_1, r_1)$ at a point x_*. Let φ_0 be an upper support function for u_0 at x_* and φ_1 a lower support function for u_1 at x_* so that for some very small ε the inequalities $\mathcal{M}[\varphi_1] \geq H_0 - \varepsilon$ and $\mathcal{M}[\varphi_0] \leq H_0 + \varepsilon$ hold. Then (2.3) and (2.4) imply

$$\begin{aligned} L_{\varphi_0,\varphi_1}(\varphi_1 - \varphi_0) &\geq -2\varepsilon - |C[\varphi_0, \varphi_1]||\varphi_1(x_*) - \varphi_0(x_*)| \\ &= -2\varepsilon - |C[\varphi_0, \varphi_1]||u_1(x_*) - u_0(x_*)|. \end{aligned}$$

Choose α in the definition of w so that $L_{\varphi_0,\varphi_1} w > 0$. The inequalities $h(x_*) \geq 0$ and $u_1(x_*) - u_0(x_*) \leq 0$ imply that $|u_1(x_*) - u_0(x_*)| \leq \delta |w(x_*)|$. As $w(x_1) = 0$, by choosing r_1 small enough we make the inequality

$$|C[\varphi_0, \varphi_1]||u_1(x_*) - u_0(x_*)| < \varepsilon.$$

hold. Then $f := (\varphi_1 - \varphi_0) + \delta w$ will satisfy

$$\begin{aligned} L_{\varphi_0,\varphi_1} f(x_*) &= L_{\varphi_0,\varphi_1}(\varphi_1 - \varphi_0)(x_*) + \delta L_{\varphi_0,\varphi_1} w(x_*) \\ &\geq -2\varepsilon - |C[\varphi_0, \varphi_1]||u_1(x_*) - u_0(x_*)| + \delta L_{\varphi_0,\varphi_1} w(x_*) \qquad (2.5) \\ &\geq -3\varepsilon + \delta L_{\varphi_0,\varphi_1} w(x_*) > 0 \end{aligned}$$

provided ε was chosen small enough. But as φ_0 is an upper support function for u_0 at x_* and φ_1 is a lower support function for u_1 at x_* the function $f = (\varphi_1 - \varphi_0) + \delta w$ will have a local maximum at x_* because $h = (u_1 - u_0) + \delta w$ has a maximum at x_*. But then $Df(x_*) = 0$ and $D^2 f(x_*)$ is negative semi-definite and $\left[A^{ij}[\varphi_0, \varphi_1]\right]$ is positive definite which implies $L_{\varphi_0,\varphi_1} f(x_*) \leq 0$. We have arrived at a contradiction to the inequality (2.5) and this completes the outline of the proof of Theorem 2.4.

The proof is more complicated than this outline indicates for two reasons. First to show $L_{\varphi_0,\varphi_1} w(x_*) > 0$ and $|C[\varphi_0, \varphi_1]||u_1(x_*) - u_0(x_*)| < \varepsilon$ we need bounds on $B^i[\varphi_0, \varphi_1]$, and $C[\varphi_0, \varphi_1]$. But the definitions of $B^i[\varphi_0, \varphi_1]$, and $C[\varphi_0, \varphi_1]$ involve the Hessians $D^2\varphi_0$ and $D^2\varphi_1$ and therefore we need to find

a priori bounds for $|D^2\varphi_0|(x_*)$ and $|D^2\varphi_1|(x_*)$. This is done in Lemma 2.10 and its Corollary 2.11 by using that at x_* the function $f = (\varphi_1 - \varphi_0) + \delta w$ has a maximum at x_* so the Hessian $D^2 f(x_*)$ will be negative semi-definite. Using $D^2 f(x_*) \leq 0$ and the one–sided bound $D^2\varphi_1 \geq -C_{\mathbf{S}} I$ gives a lower bound $D^2\varphi_0 \geq D^2\varphi_1 + \delta D^2 w$ for $D^2\varphi_0$ at x_*. The operator $\mathcal{M}$ is uniformly elliptic so the lower bound on $D^2\varphi(x_*)$ and $\mathcal{M}[\varphi_0] \leq H_0 + \varepsilon$ implies an upper bound on $D^2\varphi_0$. Then again using that $D^2 f = (\varphi_1 - \varphi_0) + \delta w \leq 0$ so that $D^2\varphi_1 \leq D^2\varphi_0 - \delta D^2 w$ we get an upper bound on $D^2\varphi_1$. This gives the required (two–sided) bounds on $D^2\varphi_0$ and $D^2\varphi_1$ in terms of δ and α (which appears in the definition of w). Which brings us to the second problem. In the outline above the choices of α, δ, r_1 etc. depend on the bounds of $B^i[\varphi_0, \varphi_1]$, $C[\varphi_0, \varphi_1]$ which in turn depend on α, δ, r_1 etc. Thus at least formally the argument is circular. However when the relations required between the various parameters are written out explicitly it is possible, with some care, to make the choices so that the argument works.

2.2. **Reduction to a Standard Setup.** We first note if we can prove the theorem under the assumption that Ω is convex then we can express an arbitrary connected domain as a union of convex subdomains and deduce $u_0 \equiv u_1$ in the entire domain by doing a standard “analytic continuation” argument. Thus assume Ω is convex. While in some applications it is convenient to work with the constant H_0 in Theorem 2.4 being non-zero, by replacing $b(x, r, p)$ by $b(x, r, p) - H_0$ it is enough to prove the result with $H_0 = 0$. (As we are assuming as part of the definition of uniform ellipticity that $|b(x, r, p)| \leq C_{\mathbf{E}}$ we may have to replace the constant $C_{\mathbf{E}}$ by $C_{\mathbf{E}} + |H_0|$.) As we will be using the same hypothesis in several of the lemmata below it is convenient to give a name to data x_0, x_1, r_0, r_1, α, δ, φ_0, φ_1 that satisfies these hypotheses and at the same time make some normalizations that will simplify notation. We will choose x_0 as above so that the closed ball $\overline{B}(x_0, 2r_0)$ meets $K := \{x \in \Omega : u_0(x) = u_1(x)\}$ at exactly one point x_1 and this point is on $\partial B(x_0, 2r_0)$.

By translating the coordinates we can assume $x_0 = 0$, the origin of the coordinate system. With these simplifications we make the following definition.

Definition 2.6. The data x_0, x_1, x_*, r_0, r_1, α, δ, φ_0, φ_1 will be said to satisfy the **standard setup** if and only if

1. $x_0 = 0$ and x_1 is a point with $\|x_1 - x_0\| = \|x_1\| = 2r_0$.
2. The positive numbers r_0, r_1 satisfy
$$r_1 \leq r_0 < 3r_0 \leq 1.$$
3. The number α is positive and we use it to define the **comparison function**
$$w(x) := \|x\|^{-\alpha}.$$
4. The number δ is positive, the point x_* is in the interior of $B(x_1, r_1)$, the functions φ_0 and φ_1 are C^2 and defined in some neighborhood of

x_* (but not necessarily in all of $B(x_1, r_1)$) and

$$f := (\varphi_1 - \varphi_0) + \delta w$$

has a local maximum at x_*.

5. The function φ_1 satisfies the one–sided Hessian bound

$$C_{\mathbf{S}} I \leq D^2 \varphi_1(x_*) \tag{2.6}$$

6. If $U \subset \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ is the set in Definition 2.2 then φ_0 is U–admissible and satisfies $\mathcal{M}[\varphi_0](x_*) \leq C_{\mathbf{E}}$. As we are assuming that $H_0 = 0$ and from the definition of being uniformly elliptic $|b(x, \varphi_0, D\varphi_0)| \leq C_{\mathbf{E}}$, this implies that at the point x_*

$$\sum_{i,j} a^{ij}(x_*, \varphi_0, D\varphi_0) D_{ij} \varphi_0(x_*) \leq 2C_{\mathbf{E}} \tag{2.7}$$

□

2.3. **Some Calculus Lemmata.**

Lemma 2.7. *Let $w := \|x\|^{-\alpha}$ where $\alpha > 0$ and $3r_0 \leq 1$ are as in the standard setup. Then w is a decreasing function of $\|x\|$ and for $r_0 \leq \|x\| \leq 3r_0$ the inequalities*

$$\|Dw\|(x) \leq \alpha \|x\|^{-(\alpha+2)} \leq \alpha r_0^{-(\alpha+2)} \tag{2.8}$$

and

$$-\alpha r_0^{-(\alpha+2)} I \leq D^2 w \leq \alpha(\alpha+2) r_0^{-(\alpha+2)} I \tag{2.9}$$

hold.

Proof. By direct calculation

$$\begin{aligned} D_i w &= -\alpha \|x\|^{-(\alpha+2)} x^i, \\ D_{ij} w &= \alpha(\alpha+2)\|x\|^{-(\alpha+4)} x^i x^j - \alpha \|x\|^{-(\alpha+2)} \delta_{ij}. \end{aligned} \tag{2.10}$$

From this it follows $\|Dw\|(x) = \alpha\|x\|^{-(\alpha+2)}\|x\| \leq \alpha\|x\|^{-(\alpha+2)} \leq \alpha r_0^{-(\alpha+2)}$ as $r_0 \leq \|x\| \leq 1$. The matrix $[x^i x^j]$ satisfies $0 \leq [x^i x^j] \leq \|x\|^2 I$ which, along with (2.10), implies the Hessian bounds (2.9) □

The following lemma is not used in the proof of the analytic maximum principle, but is important in the proof of the geometric version of the maximum principle.

Lemma 2.8. *Let w be as in the Lemma 2.7. If $f = \varphi_1 - \varphi_0 + \delta w$ has a local maximum at $x_* \in B(x_1, r_1)$ with $r_1 \leq r_0$ and $\|x_*\| \geq r_0$ then*

$$\|D\varphi_1(x_*) - D\varphi_0(x_*)\| \leq \delta \alpha r_0^{-(\alpha+2)}.$$

Proof. At a local maximum $Df(x_*) = 0$. Using the estimate on $\|Dw\|$ from Lemma 2.7, the equation $Df(x_*) = 0$ implies $\|D\varphi_1(x_*) - D\varphi_0(x_*)\| = \delta\|Dw(x_*)\| \leq \delta\alpha r_0^{-(\alpha+2)}$. □

Lemma 2.9. *Let A and B be $n \times n$ symmetric matrices with A positive definite and let c_1, c_2, c_3 and c_4 be positive constants so that*

$$\frac{1}{c_1} I \leq A \leq c_2 I, \quad B \geq -c_3 I, \quad \operatorname{trace}(AB) \leq c_4$$

then

$$B \leq c_1\big((n-1)c_2c_3 + c_4\big)I.$$

Proof. By doing an orthogonal change of basis we can assume B is diagonal, say $B = \operatorname{Diag}(\beta^1, \ldots, \beta^n)$. Let $A = [a^{ij}]$. Then

$$\operatorname{trace}(AB) = a^{11}\beta^1 + a^{22}\beta^2 + \cdots + a^{nn}\beta^n \leq c_4.$$

Solving this inequality for β^1 and using that the bounds on A and B imply the diagonal elements of A satisfy $(1/c_1) \leq a^{ii} \leq c_2$ and $\beta^i \geq -c_3$, so that in particular $-a^{ii}\beta^i \leq c_2c_3$ and $1/a^{11} \leq c_1$. Thus

$$\begin{aligned}
\beta^1 &\leq \frac{1}{a^{11}}(-a^{22}\beta^2 - \cdots - a^{nn}\beta^n + c_4) \\
&\leq \frac{1}{a^{11}}\big((n-1)c_2c_3 + c_4\big) \\
&\leq c_1\big((n-1)c_2c_3 + c_4\big),
\end{aligned}$$

and a similar calculation shows $\beta^i \leq c_1\left((n-1)c_2c_3 + c_4\right)$ for all i. But as the β^i are the eigenvalues of B this implies $B \leq \left((n-1)c_2c_3 + c_4\right) I$ and completes the proof. ☐

Lemma 2.10. *Let x_0, x_1, x_*, r_0, r_1, α, δ, φ_0, φ_1 be as in the standard setup. Then*

$$|D^2\varphi_0|(x_*) + |D^2\varphi_1|(x_*) \leq 2\left(C_{\mathbf{E}}^2\big((n-1)\big(C_{\mathbf{S}} + \delta\alpha r_0^{-(\alpha+2)}\big) + 2\big) + \delta\alpha r_0^{-(\alpha+2)}\right). \tag{2.11}$$

Proof. The Hessian of f is $D^2 f = (D^2\varphi_1 - D^2\varphi_0) + \delta D^2 w$. Part of the standard setup is that the function $f = (\varphi_1 - \varphi_0) + \delta w$ has a local maximum at x_*, and at a local maximum the Hessian satisfies $D^2 f \leq 0$. Therefore at x_* we have $D^2\varphi_1 - D^2\varphi_0 + \delta D^2 w \leq 0$. Solving for $D^2\varphi_0$ in this inequality and using the inequality $-D^2 w(x) \leq \alpha\|x\|^{-(\alpha+2)} I$ from Lemma 2.7 and the inequality $D^2\varphi_1(x_*) \geq -C_{\mathbf{S}} I$ from the standard setup we find that at x_*,

$$D^2\varphi_0(x_*) \geq D^2\varphi_1(x_*) + \delta D^2 w(x_*) \geq -\left(C_{\mathbf{S}} + \delta\alpha r_0^{-(\alpha+2)}\right) I.$$

This gives a lower bound on $D^2\varphi_0(x_*)$. Define symmetric matrices $A := [a^{ij}(x_*, \varphi_0, D\varphi_0)]$ and $B = D^2\varphi_0(x_*)$. Then as $\mathcal{M}$ is uniformly elliptic the inequalities $(1/C_{\mathbf{E}})I \leq A \leq C_{\mathbf{E}} I$ hold and we have just derived the lower bound $B \geq -\left(C_{\mathbf{S}} + \delta\alpha r_0^{-(\alpha+2)}\right) I$. By inequality (2.7) of the standard set

up the inequality $\operatorname{trace}(AB) \le 2C_{\mathbf{E}}$ holds. Thus by the last lemma,

$$\begin{aligned} D^2\varphi_0(x_*) &\le C_{\mathbf{E}}\Big((n-1)C_{\mathbf{E}}\big(C_{\mathbf{S}} + \delta\alpha r_0^{-(\alpha+2)}\big) + 2C_{\mathbf{E}}\Big)\, I \\ &= C_{\mathbf{E}}^2\Big((n-1)\big(C_{\mathbf{S}} + \delta\alpha r_0^{-(\alpha+2)}\big) + 2\Big)\, I. \end{aligned}$$

Again we use that $D^2 f(x_*) = D^2\varphi_1(x_*) - D^2\varphi_0(x_*) + \delta D^2 w(x_*) \le 0$, but this time solve for $D^2\varphi_1(x_*)$. Using the upper bound we have just derived for $D^2\varphi_0(x_*)$ and the bound $-D^2 w \le \alpha r_0^{-(\alpha+2)} I$ of inequality (2.9) we obtain,

$$D^2\varphi_1(x_*) \le D^2\varphi_0(x_*) - \delta D^2 w(x_*) \le \beta I,$$

where

$$\beta = C_{\mathbf{E}}^2\big((n-1)\big(C_{\mathbf{S}} + \delta\alpha r_0^{-(\alpha+2)}\big) + 2\big) + \delta\alpha r_0^{-(\alpha+2)}. \tag{2.12}$$

We now have upper and lower bounds on both $D^2\varphi_0(x_*)$ and $D^2\varphi_1(x_*)$. The largest of the constants to appear in these bounds is the constant on the right hand side of the upper bound on $D^2\varphi_1(x_*)$. Therefore,

$$-\beta I \le D^2\varphi_0(x_*), D^2\varphi_1(x_*) \le \beta I,$$

with β given by (2.12. But if a symmetric matrix $S = [S^{ij}]$ satisfies $-\beta I \le S \le \beta I$ the entries satisfy $|S^{ij}| \le \beta$. The inequality (2.11) now follows. □

Corollary 2.11. *Assume in addition to the hypotheses of the last lemma that δ satisfies*

$$\delta \le \overline{\delta}(\alpha) := \frac{r_0^{\alpha+2}}{\alpha}. \tag{2.13}$$

Then

$$|D^2\varphi_0|(x_*) + |D^2\varphi_1|(x_*) \le C_{\mathbf{H}},$$

where

$$C_{\mathbf{H}} := 2\left(C_{\mathbf{E}}^2\big((n-1)\big(C_{\mathbf{S}} + 1\big) + 2\big) + 1\right) \tag{2.14}$$

only depends on $C_{\mathbf{E}}$, $C_{\mathbf{S}}$ and the dimension n.

Proof. The bound (2.13) implies $\delta\alpha r_0^{-(\alpha+2)} \le 1$ and hence the result follows from (2.11). □

Lemma 2.12. *Let $\mathcal{M}$ and U be as in the statement of Theorem 2.4. Let φ_0 and φ_1 be U–admissible C^2 functions defined on some open subset of Ω. Then*

$$\begin{aligned} \mathcal{M}[\varphi_1] - \mathcal{M}[\varphi_0] = &\sum_{i,j} A^{ij}[\varphi_0,\varphi_1] D_{ij}(\varphi_1 - \varphi_0) + \sum_i B^i[\varphi_0,\varphi_1] D_i(\varphi_1 - \varphi_0) \\ &+ C[\varphi_0,\varphi_1](\varphi_1 - \varphi_0) \end{aligned}$$

where the coefficients $A^{ij}[\varphi_0,\varphi_1] = A^{ji}[\varphi_0,\varphi_1]$ satisfy the estimates

$$\frac{1}{C_{\mathbf{E}}}\|\xi\|^2 \le \sum_{i,j} A^{ij}[\varphi_0,\varphi_1]\xi_i\xi_j \le C_{\mathbf{E}}\|\xi\|^2, \tag{2.15}$$

so that

$$\frac{1}{C_{\mathbf{E}}} \le A^{ii}[\varphi_0,\varphi_1] \le C_{\mathbf{E}}, \quad \textit{and} \quad |A^{ij}[\varphi_0,\varphi_1]| \le C_{\mathbf{E}}. \tag{2.16}$$

The coefficients $B^i[\varphi_0,\varphi_1]$, *and* $C[\varphi_0,\varphi_1]$ *satisfy*

$$|B^i[\varphi_0,\varphi_1]|,\ |C[\varphi_0,\varphi_1]| \le n^2 C_{\mathbf{E}}(|D^2\varphi_0| + |D^2\varphi_1| + 1). \tag{2.17}$$

Proof. One of the hypotheses of Theorem 2.4 is that the fibers U_x are convex. Thus if both φ_0 and φ_1 are U–admissible so is $\varphi_t := (1-t)\varphi_0 + t\varphi_1$ for $0 \le t \le 1$. Therefore we can compute

$$\begin{aligned}
\mathcal{M}[\varphi_1] - \mathcal{M}[\varphi_0] &= \int_0^1 \frac{d}{dt}\mathcal{M}[\varphi_t]\,dt \\
&= \sum_{i,j} \int a^{ij}(x,\varphi_t,D\varphi_t)\,dt\, D_{ij}(\varphi_1-\varphi_0) \\
&\quad + \sum_{i,j,k} \int_0^1 \frac{\partial a^{ij}}{\partial p^k}(x,\varphi_t,D\varphi_t) D_{ij}\varphi_t\,dt\, D_k(\varphi_1-\varphi_0) \\
&\qquad + \sum_i \int_0^1 \frac{\partial b}{\partial p^i}(x,\varphi_t,D\varphi_t)\,dt\, D_i(\varphi_1-\varphi_0) \\
&\quad + \sum_{i,j} \int_0^1 \frac{\partial a^{ij}}{\partial r}(x,\varphi_t,D\varphi_t) D_{ij}\varphi_t\,dt\,(\varphi_1-\varphi_0) \\
&\qquad + \int_0^1 \frac{\partial b}{\partial r}(x,\varphi_t,D\varphi_t)\,dt\,(\varphi_1-\varphi_0) \\
&= \sum_{i,j} A^{ij}[\varphi_0,\varphi_1] D_{ij}(\varphi_1-\varphi_0) + \sum_i B^i[\varphi_0,\varphi_1] D_i(\varphi_1-\varphi_0) \qquad (2.18)\\
&\quad + C[\varphi_0,\varphi_1](\varphi_1-\varphi_0),
\end{aligned}$$

where, by re-indexing some of the sums involved, we find the coefficients A^{ij}, B^i and C have the formulas

$$A^{ij}[\varphi_0,\varphi_1] := \int_0^1 a^{ij}(x,\varphi_t,D\varphi_t)\,dt, \tag{2.19}$$

$$B^i[\varphi_0,\varphi_1] := \int_0^1 \left(\sum_{j,k} \frac{\partial a^{jk}}{\partial p^i}(x,\varphi_t,D\varphi_t) D_{jk}\varphi_t + \frac{\partial b}{\partial p^i}(x,\varphi_t,D\varphi_t) \right) dt, \tag{2.20}$$

$$C[\varphi_0,\varphi_1] := \int_0^1 \left(\sum_{i,j} \frac{\partial a^{ij}}{\partial r}(x,\varphi_t,D\varphi_t) D_{ij}\varphi_t + \frac{\partial b}{\partial r}(x,\varphi_t,D\varphi_t) \right) dt. \tag{2.21}$$

Then (2.15) follows from (2.1) by integration. The inequalities (2.16) are algebraic consequences of (2.15). From $\varphi = (1-t)\varphi_0 + t\varphi_1$ for $t \in [0,1]$ we

get,

$$|D^2\varphi_t| \le (1-t)|D^2\varphi_0| + t|D^2\varphi_1| \le |D^2\varphi_0| + |D^2\varphi_1|,$$

so that if $C_{\mathbf{E}}$ is the constant in (2.2),

$$\left|\int_0^1 \frac{\partial a^{jk}}{\partial p^i}(x,\varphi_t,D\varphi_t)D_{jk}\varphi_t\,dt\right| \le C_{\mathbf{E}}(|D^2\varphi_0| + |D^2\varphi_1|),$$

$$\left|\int_0^1 \frac{\partial b}{\partial p^i}(x,\varphi_t,D\varphi_t)\,dt\right| \le C_{\mathbf{E}}.$$

Using this in the formula (2.20) defining $B^i[\varphi_0,\varphi_1]$ and the inequality $1 \le n^2$ we obtain,

$$|B^i[\varphi_0,\varphi_1]| \le C_{\mathbf{E}}(n^2(|D^2\varphi_0| + |D^2\varphi_1|) + 1) \le n^2 C_{\mathbf{E}}(|D^2\varphi_0| + |D^2\varphi_1| + 1)$$

as required. The derivation of the bound on $C[\varphi_0,\varphi_1]$ is identical. □

Corollary 2.13. *Let x_0, x_1, x_*, r_0, r_1, α, δ, φ_0, φ_1 be as in the standard setup and assume $\delta \le \overline{\delta}(\alpha)$, as given in (2.13). Then*

$$|B^i[\varphi_0,\varphi_1]|(x_*),\ |C[\varphi_0,\varphi_1]|(x_*) \le n^2 C_{\mathbf{E}}(C_{\mathbf{H}}+1),$$

and

$$\|B[\varphi_0,\varphi_1]\| \le n^3 C_{\mathbf{E}}(C_{\mathbf{H}}+1), \tag{2.22}$$

where $C_{\mathbf{H}}$ is as in equation (2.14).

Proof. This follows by using the estimates (2.17) and (2.11) in the last lemma, and in the estimate for $\|B[\varphi_0,\varphi_1]\|$ the inequality $n^{5/2} \le n^3$ was used. □

Lemma 2.14. *Let x_0, x_1, x_*, r_0, r_1, α, δ, φ_0, φ_1 be as in the standard setup and set*

$$\alpha = \overline{\alpha} := -2 + C_{\mathbf{E}}\left(1 + nC_{\mathbf{E}} + n^3 C_{\mathbf{E}}(C_{\mathbf{H}}+1)\right) \tag{2.23}$$

(This makes α a solution to $\big((\alpha+2)/C_{\mathbf{E}} - nC_{\mathbf{E}} - n^3C_{\mathbf{E}}(C_{\mathbf{H}}+1)\big) = 1$). Also assume δ satisfies the inequality (2.13), and let L_{φ_0,φ_1} be as defined in Equation (2.4). Then

$$L_{\varphi_0,\varphi_1} w(x_*) \ge 1.$$

Proof. We first note for future reference that $n, C_{\mathbf{E}} \ge 1$, $C_{\mathbf{H}} > 0$ implies $\overline{\alpha} \ge -2+3 = 1$. Under the hypotheses of the lemma, the bounds in (2.16) on $A^{ij}[\varphi_0,\varphi_1]$ and the bound (2.22) on $\|B[\varphi_0,\varphi_1]\|\,(x_*)$ hold. We will also use the bound $\|Dw\|(x) \le \alpha\|x\|^{-(\alpha+2)}$ of Lemma 2.7. To simplify notation we assume all functions are evaluated at $x = x_*$ and drop the subscript of $*$.

Using (2.8) and the explicit formula (2.10) for $D_{ij}w$ we have,

$$\begin{aligned}
L_{\varphi_0,\varphi_1} w(x_*) &= \sum_{i,j} A^{ij}[\varphi_0,\varphi_1] D_{ij} w + \sum_i B^i[\varphi_0,\varphi_1] D_i w \\
&= \alpha(\alpha+2)\|x\|^{-(\alpha+4)} \sum_{i,j} A^{ij}[\varphi_0,\varphi_1] x^i x^j \\
&\qquad - \alpha\|x\|^{-(\alpha+2)} \sum_i A^{ii}[\varphi_0,\varphi_1] + \sum_i B^i[\varphi_0,\varphi_1] D_i w \\
&\geq \alpha(\alpha+2)\|x\|^{-(\alpha+4)} \frac{\|x\|^2}{C_{\mathbf{E}}} - \alpha\|x\|^{-(\alpha+2)} n C_{\mathbf{E}} - \|B[\varphi_0,\varphi_1]\|\,\|Dw\| \\
&\geq \alpha\|x\|^{-(\alpha+2)} \left(\frac{\alpha+2}{C_{\mathbf{E}}} - nC_{\mathbf{E}} \right) - n^3 C_{\mathbf{E}}(C_{\mathbf{H}}+1)\alpha\|x\|^{-(\alpha+2)} \\
&= \alpha\|x\|^{-(\alpha+2)} \left(\frac{\alpha+2}{C_{\mathbf{E}}} - nC_{\mathbf{E}} - n^3 C_{\mathbf{E}}(C_{\mathbf{H}}+1) \right) \\
&= \alpha\|x\|^{-(\alpha+2)} \;\geq\; 1,
\end{aligned}$$

where the last couple of steps hold because of the choice of $\alpha = \overline{\alpha}$ (so that also $\alpha \geq 1$), and $\|x\|^{-(\alpha+2)} \geq 1$ as $\|x\| \leq 3r_0 \leq 1$. This completes the proof. $\square$

2.4. **Proof of the Maximum Principle.** We follow the outline given in §2.1 and assume that we have already made the normalizations given in §2.2. We then choose α as in Lemma 2.14:

$$\alpha = \overline{\alpha} = -2 + C_{\mathbf{E}}\left(1 + nC_{\mathbf{E}} + n^3 C_{\mathbf{E}}(C_{\mathbf{H}}+1)\right)$$

This is a formula for α in terms of just the parameters $C_{\mathbf{E}}$, $C_{\mathbf{S}}$ and the dimension n (recall the definition of $C_{\mathbf{H}}$ given in equation (2.14)). Let

$$r_1 = \min\left\{ r_0,\ \frac{1}{4\left(n^2 C_{\mathbf{E}}(C_{\mathbf{H}}+1)\right)\left(\alpha r_0^{-(\alpha+2)}\right)} \right\}$$

Thus, keeping in mind the definition of α, r_1 only depends on r_0, $C_{\mathbf{E}}$, $C_{\mathbf{S}}$, and n. Let S', S'' as in §2.1 and let $h := (u_1 - u_0) + \delta(w - w(x_1))$ where we now determine how to choose δ. The function $u_1 - u_0$ is negative and upper semi-continuous on the compact set S' and so there is a $\delta_1 > 0$ such that $(u_1 - u_0) + \delta_1(w - w(x_1)) < 0$ on S'. Let

$$\delta := \min\{\delta_1, \overline{\delta}(\alpha)\},$$

where $\overline{\delta}(\alpha)$ is given by (2.13) (so δ is defined just in terms of $C_{\mathbf{E}}$, $C_{\mathbf{S}}$, n, r_0, and δ_1). As $\delta \leq \delta_1$ we have $h < 0$ on S'. The function w is a decreasing function of $\|x - x_0\| = \|x\|$ so the function h will be negative on the set S''. Therefore h is negative on all of $\partial B(x_1, r_1) = S' \cup S''$. But x_1 was chosen so that $u_1(x_1) = u_0(x_1)$ and thus $h(x_1) = (u_1(x_1) - u_0(x_1)) + \delta(w(x_1) - w(x_1)) = 0$. Hence h has an local maximum at some interior point x_* of $B(x_1, r_1)$.

Choose $\varepsilon > 0$ so that

$$\varepsilon < \min\left\{\frac{\delta}{4}, C_{\mathbf{E}}\right\}.$$

Recall that we are assuming $H_0 = 0$. Let φ_0 be an upper support function for u_0 at x_* and φ_1 a lower support function for u_1 at x_* so that $\mathcal{M}[\varphi_0] \leq \varepsilon$, $\mathcal{M}[\varphi_1] \geq -\varepsilon$ and the one–sided Hessian bound (2.6) holds. Using that φ_0 and φ_1 are upper and lower support functions at x_* we see $f := (\varphi_1 - \varphi_0) + \delta w$ has a local maximum at x_*. Therefore x_0, x_1, x_*, r_0, r_1, α, δ, φ_0, φ_1 satisfy all the conditions of the standard setup together with the hypotheses of Lemma 2.14. Using formula (2.3), the definition (2.4) of L_{φ_0,φ_1}, and the choice of ε

$$\begin{aligned}-\frac{\delta}{2} \leq -2\varepsilon &\leq \mathcal{M}[\varphi_1](x_*) - \mathcal{M}[\varphi_0](x_*)\\ &= L_{\varphi_0,\varphi_1}(\varphi_1 - \varphi_0)(x_*) + C[\varphi_0,\varphi_1](\varphi_1 - \varphi_0)(x_*),\end{aligned}$$

so that

$$L_{\varphi_0,\varphi_1}(\varphi_1 - \varphi_0)(x_*) \geq -\frac{\delta}{2} - |C[\varphi_0,\varphi_1]||\varphi_1(x_*) - \varphi_0(x_*)|.$$

By Lemma 2.14 $L_{\varphi_0,\varphi_1} w(x_*) \geq 1$. Thus for $f = (\varphi_1 - \varphi_0) + \delta w$ we use the last displayed inequality, the bound of Corollary 2.13 on $|C[\varphi_0,\varphi_1]|$, and the equality $u_i(x_*) = \varphi_i(x_*)$ to compute

$$\begin{aligned}L_{\varphi_0,\varphi_1} f(x_*) &\geq L_{\varphi_0,\varphi_1}(\varphi_1 - \varphi_0)(x_*) + \delta \geq \frac{\delta}{2} - |C[\varphi_0,\varphi_1]|(\varphi_1(x_*) - \varphi_0(x_*)|\\ &\geq \frac{\delta}{2} - n^2 C_{\mathbf{E}}(C_{\mathbf{H}} + 1)|u_1(x_*) - u_0(x_*)|. \qquad (2.24)\end{aligned}$$

We now derive an estimate on $|u_1(x_*) - u_0(x_*)|$. From the hypotheses of the theorem $u_1(x_*) - u_0(x_*) \leq 0$. The function $h = (u_1 - u_0) + \delta(w - w(x_1))$ has its maximum in the ball $B(x_1, r_1)$ at the point x_* and $h(x_1) = 0$ and thus $h(x_*) = (u_1(x_*) - u_0(x_*)) + \delta(w(x_*) - w(x_1)) \geq 0$. Therefore, using the bound (2.8) on $\|Dw\|$, we have

$$|u_1(x_*) - u_0(x_*)| \leq \delta|w(x_*) - w(x_1)| \leq \delta\|x_* - x_1\|\alpha r_0^{-(\alpha+2)} \leq \delta\alpha r_0^{-(\alpha+2)} r_1.$$

Using this in (2.24) along with the definition of r_1 gives

$$L_{\varphi_0,\varphi_1} f(x_*) \geq \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4} > 0.$$

But from the first and second derivative tests of calculus $Df(x_*) = 0$ and $D^2 f(x_*) \leq 0$ so

$$L_{\varphi_0,\varphi_1} f(x_*) = \sum_{i,j} A^{ij}[\varphi_0,\varphi_1] D_{ij} f(x_*) \leq 0.$$

This contradicts $L_{\varphi_0,\varphi_1} f(x_*) > 0$ and completes the proof $u_0 \equiv u_1$.

We now set $u := u_0 = u_1$ and show u is locally a $C^{1,1}$ function, i.e. that $Du(x)$ exists for all $x_0 \in \Omega$ and it locally satisfies a Lipschitz condition $\|Du(x_1) - Du(x_0)\| \leq C\|x_1 - x_0\|$ for some $C \geq 0$. First, as $u = u_1 = u_0$

is both upper and lower semi-continuous it is continuous. As $u_0 = u_1$ the function $u_1 - u_0 \equiv 0$ has a local maximum at every point of its domain Ω. Letting x be any point in Ω, let φ_0 be a C^2 upper support function for u_0 at x with $\mathcal{M}[\varphi_0] \le C_{\mathbf{E}}$ at x and let φ_1 be a C^2 lower support for u_1 at x which satisfies the one–sided Hessian bound $D^2\varphi_1(x) \ge -C_{\mathbf{S}}I$. Then $\varphi_1 - \varphi_0$ has a local maximum at x and a simplified version of the proof of Lemma 2.10 (with $\delta = 0$) shows there is a constant $C'_{\mathbf{H}}$ independent of x so that $-C'_{\mathbf{H}}I \le D^2\varphi_0, D^2\varphi_1 \le C'_{\mathbf{H}}I$. (Going through the proof of Lemma 2.10 with $\delta = 0$ shows that $C'_{\mathbf{H}} = C^2_{\mathbf{E}}\big((n-1)C_{\mathbf{S}} + 2\big)$ works.)

The following is certainly known, but as we have not found an explicit reference we include a short proof.

Lemma 2.15. *Let v be a continuous function on the convex open set Ω and assume that v satisfies $D^2v \ge -CI$ in the sense of support functions for some $C \in \mathbf{R}$. (That is for every $x \in \Omega$ there is a C^2 lower support function φ to v at x so that $D^2\varphi \ge -CI$ near x.) Then for each $x_0 \in \Omega$ there is a vector $a \in \mathbf{R}^n$ so that for all $x \in \Omega$*

$$v(x) \ge v(x_0) + \langle x - x_0, a\rangle - \frac{1}{2}C\|x - x_0\|^2.$$

If v is differentiable at x_0 then $a = Dv(x_0)$.

Proof. We first give the proof in the one dimensional case. Then $\Omega \subseteq \mathbf{R}$ is an interval. Let φ be a lower support function to v at x_0 that satisfies $D^2\varphi = \varphi'' \ge -C$ near x_0 and let $a = D\varphi(x_0)$. We claim $v(x) \ge v(x_0) + a(x-x_0) - (C/2)(x-x_0)^2$. Let $f(x) := v(x_0) + a(x-x_0) - (C/2)(x-x_0)^2$. As $\varphi(x_0) = f(x_0)$, $\varphi'(x_0) = f'(x_0)$ and $\varphi''(x) \ge -C = f''(x)$ we have an interval about x_0 so that $v(x) \ge \varphi(x) \ge f(x)$. Assume, toward a contradiction, there is an x_1 so that $v(x_1) - f(x_1) < 0$. Then the function $v_1 := v - f$ will satisfy $v_1'' \ge 0$ in the sense of support functions and $v_1(x_0) = 0$, $v_1(x_1) \le 0$ and the function $v_1(x) \ge 0$ for x near x_0 so the function v_1 will have an maximum at a point x_* between x_0 and x_0. Let v_0 be the constant function $v_0(x) = v_1(x_*)$. Then $v_1 \le v_0$, $v_1(x_*) - v_0(x_*)$, and $v_1'' \ge 0$ (in the sense of support functions) and $v_0'' = 0$ (in the strong sense) so the one variable case of of the linear maximum principle (which is easy to verify) implies $v_0(x) = v_1(x)$ for x between x_0 and x_1. As $v_1(x_0) = 0$ and v_0 is constant this implies $v_1(x) = v(x) - f(x)$ for all x between x_0 and x_1. This in particular implies $0 = v(x_1) = f(x_1)$ which contradicts the assumption $v(x_1) - f(x_1) < 0$ and completes the proof in the one dimensional case.

We return to the general case. Let φ be a lower support function for v at x_0 that satisfies $D^2\varphi \ge -CI$ near x_0 and let $a := D\varphi(x_0)$. For any unit vector $b \in \mathbf{R}^n$ let $v_b(t) = v(x_0+tb) - t\langle b, a\rangle$. Then a lower support ψ function to v at $x_0 + t_0b$ that satisfies $D^2\psi \ge -CI$ yields the lower support function $\psi_b(t) := \psi(x_0 + tb)$ to v_b at t_0 that satisfies $\psi_b''(t) \ge -C$ near t_0. Thus v_b satisfies the one variable version of the result and so $v_b(t) = v(x_0 + tb) \ge v(x_0) - t\langle b, a\rangle - (C/2)t^2$. But as Ω is convex every point of Ω can be written

as $x = x_0 + tb$ for some t and some unit vector and so the multidimensional case reduces to the one dimensional case. This completes the proof. $\square$

The last lemma implies at each point of Ω that u has global upper and lower support paraboloids with "opening" $2C_{\mathbf{H}}$ (that is, in the terminology of Caffarelli and Cabre [8], the second order term of the paraboloid is $\pm C_{\mathbf{H}}\|x\|^2$). It now follows that u is of class $C^{1,1}$, see for example [8, Prop 1.1 p7].

Now assume a^{ij} and b are locally of class $C^{0,\alpha}$ of all its arguments (x, r, p) for some $\alpha \in (0, 1)$ (this is in addition to the assumption that they are C^1 functions of the arguments (r, p)) and we will show $u = u_0 = u_1$ is locally $C^{2,\alpha}$. Define a linear second order differential operator by

$$\mathcal{A}_u f := \sum_{i,j} a^{ij}(x, u(x), Du(x)) D_{ij} f.$$

As Du is locally Lipschitz the functions $x \mapsto a^{ij}(x, u(x), Du(x))$ and $x \mapsto b(x, u(x), Du(x))$ are locally of class $C^{0,\alpha}$. Let B be an open ball whose closure is contained in Ω. Then by a standard existence result of the linear Schauder theory (cf. [17, Thm 6.13 p101]) gives there is a unique function v continuous on the closed ball $\overline{B}$, locally of class $C^{2,\alpha}$ in B and so that v solves the boundary value problem

$$\mathcal{A}_u v(x) = -b(x, u(x), Du(x)) + H_0 \quad \text{in } B, \qquad v = u \quad \text{on } \partial B.$$

But u solves the same boundary value problem

$$\mathcal{A}_u u(x) = -b(x, u(x), Du(x)) + H_0 \quad \text{in } B, \qquad u = u \quad \text{on } \partial B$$

but in the sense of support functions rather than in the classical sense. Thus the function $f := u - v$ will satisfy $\mathcal{A}_u f = 0$ in B in the sense of support functions and $f = 0$ on ∂B. Then $f = 0$ in B by Calabi's version of the Hopf maximum principle [9]. Therefore $u = v$ in B so that u is locally of class $C^{2,\alpha}$ in B. As every point of Ω is contained in such a ball B the function u is locally of class $C^{2,\alpha}$.

We use this as the base of an induction to prove higher regularity. Assume for some $k \geq 1$ the functions a^{ij} and b are locally $C^{k,\alpha}$ and u is locally $C^{k+1,\alpha}$. Then $x \mapsto a^{ij}(x, u(x), Du(x))$ and $x \mapsto b(x, u(x), Du(x))$ are locally $C^{k,\alpha}$. But u is a solution to the linear equation $\mathcal{A}_u u(x) = -b(x, u(x), Du(x)) + H_0$ and by the linear Schauder regularity theory (cf. [17, Thm 6.17 p104]) this implies u is locally of class $C^{k+2,\alpha}$. This completes the induction step and finishes the proof of Theorem 2.4.

3. Geometric Maximum Principles for Hypersurfaces in Lorentzian and Riemannian Manifolds

The version of the analytic maximum principle given by Theorem 2.4 is especially natural in the Lorentzian setting as C^0 spacelike hypersurfaces (Definition 3.1 below) can always be locally represented as graphs. Theorem 2.4

also applies to hypersurfaces in Riemannian manifolds that can be represented locally as graphs (see Theorem 3.10). However in the Riemannian setting there are many cases, such as horospheres, where one wants to use the maximum principle but this graph condition does not *a priori* hold. A version of the geometric maximum principle adapted to horospheres and other rough hypersurfaces is given in [21]. We now give a detailed proof of the geometric maximum principle for C^0spacelike hypersurfaces in Lorentzian manifolds.

We first state our conventions on the sign of the second fundamental form and the mean curvature. To fix our choice of signs, a Lorentzian manifold (M, g) is an n-dimensional manifold M that has semi-Riemannian g metric with signature $+, \dots, +, -$. Our results are basically local and, since every Lorentzian manifold is locally time orientable, we assume that all our Lorentzian manifolds are timed oriented and use the standard terminology of **spacetime** for a time oriented Lorentzian manifold. See [7] for background on Lorentzian geometry.

Given a smooth spacelike hypersurface $N \subset M$ we will always use the future pointing unit normal $\mathbf{n}$ which, as N is spacelike, will be timelike, $g(\mathbf{n}, \mathbf{n}) = -1$. Let ∇ be the metric connection on (M, g). Then the second fundamental form h and mean curvature H of N are defined by

$$h(X, Y) := -g(\nabla_X Y, \mathbf{n}), \qquad H := \frac{1}{n-1} \operatorname{trace}_{g\big|_N} h = \frac{1}{n-1} \sum_{i=1}^{n-1} h(e_i, e_i)$$

where X, Y are smooth vectors fields tangent to N and $e_1, , \dots, e_{n-1}$ is a smooth locally defined orthonormal frame field along N.

If (M, g) is flat Minkowski space with metric $g := (dx^1)^2 + \cdots + (dx^{n-1})^2 - (dx^n)^2$ and if N is given by a graph $x^n = f(x^1, \dots, x^{n-1})$ with $\|Df\|^2 := (\partial f/\partial x^1)^2 + \cdots (\partial f/\partial x^{n-1})^2 < 1$ then with this choice of signs, the mean curvature of N is given by

$$H = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{\partial}{\partial x^i} \left(\frac{1}{\sqrt{1 - \|Df\|^2}} \frac{\partial f}{\partial x^i} \right).$$

(Here $\partial/\partial x^n$ is in the direction of the future.) Thus the symbol of the linearization of H is positive definite. This is also the choice of sign so that if N is moved along the future pointing normal $\mathbf{n}$ then N is expanding when H is positive and shrinking when H is negative, where expanding and shrinking are measured in terms of expansion or shrinking of the area element of N. Thus very roughly H is the local "Hubble constant" of an observer on N who believes that the world-lines of rest particles are the geodesics normal to N.

In the maximum principle we wish to use the weakest natural notion of a hypersurface being spacelike. The following definition was introduced in [13] or see [7, p. 539].

Definition 3.1. A subset $N \subset M$ of the spacetime (M, g) is a C^0 **spacelike hypersurface** iff for each $p \in N$ there is a neighborhood U of p in M so that $N \cap U$ is acausal and edgeless in U. □

Remark 3.2. In this definition note that if $D(N \cap U, U)$ is the domain of dependence of $N \cap U$ in U then $D(N \cap U, U)$ is open in M and $N \cap U$ is a Cauchy hypersurface in $D(N \cap U, U)$. But a spacetime that has a Cauchy hypersurface is globally hyperbolic. Thus by replacing U by $D(N \cap U, U)$ we can assume the neighborhood U in the last definition is globally hyperbolic and that $N \cap U$ is a Cauchy surface in U. In particular, a C^0 spacelike hypersurface is a topological (in fact Lipschitz) submanifold of codimension one. □

Let (M, g) be a spacetime and let N_0 and N_1 be two C^0 spacelike hypersurfaces in (M, g) which meet at a point q. We say that N_0 is **locally to the future of** N_1 **near** q iff for some neighborhood U of p in which N_1 is acausal and edgeless, $N_0 \cap U \subset J^+(N_1, U)$, where $J^+(N_1, U)$ is the causal future of N_1 in U. In a time-dual fashion we may define what it means for N_0 to be locally to the past of N_1 near q.

Now consider a C^0 spacelike hypersurface N in a spacetime (M, g). In the context of the following definition, S is a **future support hypersurface** for N at $x_0 \in N$ iff $x_0 \in S$ and S is locally to the future of N near x_0. Time-dually, S is a **past support hypersurface** for N at $x_0 \in N$ iff $x_0 \in S$ and S is locally to the past of N near x_0.

Definition 3.3. Let N be a C^0 spacelike hypersurface in the spacetime (M, g) and H_0 a constant. Then

1. N has **mean curvature** $\le H_0$ **in the sense of support hypersurfaces** iff for all $q \in N$ and $\varepsilon > 0$ there is a C^2 future support hypersurface $S_{q,\varepsilon}$ to N at q and the mean curvature of $S_{q,\varepsilon}$ at q satisfies
$$H_q^{S_{q,\varepsilon}} \le H_0 + \varepsilon.$$
2. N has **mean curvature** $\ge H_0$ **in the sense of support hypersurfaces with one–sided Hessian bounds** iff for all compact sets $K \subseteq N$ there is a compact set $\widehat{K} \subset T(M)$ and a constant $C_K > 0$ such that for all $q \in K$ and $\varepsilon > 0$ there is a C^2 past support hypersurface $P_{q,\varepsilon}$ to N so that
   (a) The future pointing unit normal $\mathbf{n}^{P_{q,\varepsilon}}(q)$ to $P_{q,\varepsilon}$ at q is in $\widehat{K}$.
   (b) At the point q the mean curvature $H^{P_{q,\varepsilon}}$ and second fundamental form $h^{P_{q,\varepsilon}}$ of $P_{q,\varepsilon}$ satisfy
$$H_q^{P_{q,\varepsilon}} \ge H_0 - \varepsilon, \quad h_q^{P_{q,\varepsilon}} \ge -C_K g\big|_{P_{q,\varepsilon}}.$$

□

Remark 3.4. As will be seen below (Lemma 3.8 and the discussion following it) the condition that the unit normals to the support hypersurfaces $P_{q,\varepsilon}$ to N

for $q \in K$ all remain in a compact set $\widehat{K}$ is equivalent to the mean curvature operator being uniformly elliptic on the set $K \subseteq N$. Therefore the definition of N having mean curvature $\geq H_0$ in the sense of support hypersurfaces with one–sided Hessian bounds has built into it that the mean curvature operator on N is uniformly elliptic.

However in the definition of N having mean curvature $\leq H_0$ in the sense of support hypersurfaces there is no restriction on the normals to the support hypersurfaces and so the mean curvature operator need not be uniformly elliptic on N. (In the set up for the analytic maximum principle case this is equivalent to dropping the assumption that the upper support functions $\varphi_{x,\varepsilon}$ in Definition 2.3 have to be U–admissible.) Therefore in proving the geometric version of the maximum principle we will have to prove a (fortunately trivial) estimate that shows in fact the normals to the future support hypersurfaces are well behaved. $\square$

In practice it is often the case that the support hypersurfaces $S_{q,\varepsilon}$ and $P_{q,\varepsilon}$ can be chosen to be geodesic spheres through q all tangent to each other and with increasing radii in which case the one–sided bound on the second fundamental forms of the past support hypersurfaces $P_{q,\varepsilon}$ in the definition of mean curvature $\geq H_0$ in the sense of support functions with a one–sided Hessian bound holds for easy *a priori* reasons. The following proposition, whose proof just relies on the continuous dependence of solutions of ordinary differential equations on initial conditions and parameters, summarizes this. The details are left to the reader.

Proposition 3.5. *Let (M, g) be a spacetime, $r_0 > 0$ and $K \subset T(M)$ a compact set of future pointing timelike unit vectors. Assume that there is a δ so that for all $\eta \in K$, the geodesic $\gamma_\eta(t) := \exp(t\eta)$ maximizes the Lorentzian distance on the interval $[0, r_0 + \delta]$. For each $\eta \in K$ and $r > 0$ let $\pi(\eta)$ be the base point of η and set*

$$S_{\eta,r} := \{p : d(p, \exp(r\eta)) = r\}. \tag{3.1}$$

Then S_{η,r_0} contains $\pi(\eta)$ and in a neighborhood of $\pi(\eta)$ it is a smooth spacelike hypersurface whose future pointing unit normal at $\pi(\eta)$ is η. There is a uniform two–sided bound on the second fundamental forms $h^{S_{\eta,r_0}}_{\pi(\eta)}$ (or equivalently, uniform bounds on the absolute values of the principal curvatures) as η ranges over K (these bounds only depend on (M, g), K, and r_0). Thus if $\eta \in K$ and S is a smooth spacelike hypersurface that passes through $\pi(\eta)$ and locally near $\pi(\eta)$ is in the causal future of S_{η,r_0} then there is a lower bound on the second fundamental form of S at $\pi(\eta)$ that only depends on (M, g), K, and r_0. In particular if $r \geq r_0$, and $S_{\eta,r}$ is smooth near $\pi(\eta)$ then $S_{\eta,r}$ is in the causal future of S_{η,r_0} (by the reverse triangle inequality) and so there is a lower bound on the second fundamental form of $S_{\eta,r}$ only depending on (M, g), K, and r_0. $\square$

Theorem 3.6 (Lorentzian Geometric Maximum Principle)**.** *Let N_0 and N_1 be C^0 spacelike hypersurfaces in a spacetime (M, g) which meet at a point q_0, such that N_0 is locally to the future of N_1 near q_0. Assume for some constant H_0:*

1. *N_0 has mean curvature $\leq H_0$ in the sense of support hypersurfaces.*
2. *N_1 has mean curvature $\geq H_0$ in the sense of support hypersurfaces with one–sided Hessian bounds.*

Then $N_0 = N_1$ near q_0, i.e., there is a neighborhood $\mathcal{O}$ of q_0 such that $N_0 \cap \mathcal{O} = N_1 \cap \mathcal{O}$. Moreover, $N_0 \cap \mathcal{O} = N_1 \cap \mathcal{O}$ is a smooth spacelike hypersurface with mean curvature H_0.

In §3.1 below, Theorem 3.6 is reduced to the analytic maximum principle. The proof is given in §3.2.

Remark 3.7. If the metric only has finite differentiability, say g is $C^{k,\alpha}$ with $k \geq 2$ and $0 < \alpha < 1$ then, since the functions a^{ij} and b in the definition of the mean curvature operator $\mathcal{H}$ (see the proof below) depend on the first derivatives of the metric, they are of class $C^{k-1,\alpha}$. Thus the regularity part of Theorem 2.4 implies the hypersurface $N_0 \cap \mathcal{O} = N_1 \cap \mathcal{O}$ in the statement of the last theorem is $C^{k+1,\alpha}$. □

Note that hypothesis 2 is trivially satisfied if N_1 is a smooth spacelike hypersurface with mean curvature $\geq H_0$ in the usual sense. This yields a "rough-smooth" version of our geometric maximum principle which does not require any a priori Hessian estimates. This version implies the ad hoc maximum principle for the level sets of the Lorentzian Busemann function obtained in [15] to prove the Lorentzian splitting theorem (see [14] for further applications). However, the geometric maximum principle, when used in its full generality, yields a more intrinsic and conceptually simplified proof of the Lorentzian splitting theorem. See the next section for further discussion of this point and additional applications.

3.1. **Reduction to the Analytic Maximum Principle.** Let (M, g) be an n dimensional spacetime and let ∇ be the metric connection of the metric g. Then near any point q of M there is a coordinate system $(x^1, \ldots, x^n)$ so that the metric takes the form

$$g = \sum_{A,B=1}^{n} g_{AB} dx^A dx^B = \sum_{i,j=1}^{n-1} g_{ij} dx^i dx^j - (dx^n)^2 \tag{3.2}$$

and so that $\partial/\partial x^n$ is a future pointing timelike unit vector. (To construct such coordinates choose any smooth spacelike hypersurface S in M passing through q and let $(x^1, \ldots, x^{n-1})$ be local coordinate on S centered on at q. Let x^n be the signed Lorentzian distance from S. Then near q the coordinate system $(x^1, \ldots, x^n)$ is as required.) Let f be a function defined near the origin in $\mathbf{R}^{n-1}$ with $f(0) = 0$. Then define a map F_f from a neighborhood

of the origin in $\mathbf{R}^{n-1}$ to M so that in the coordinate system $(x^1,\dots,x^n)$ F_f is given by

$$F_f(x^1,\dots,x^{n-1}) = (x^1,\dots,x^{n-1}, f(x^1,\dots,x^{n-1})).$$

This parameterizes a smooth hypersurface N_f through x_0 and moreover every smooth spacelike hypersurface through x_0 is uniquely parameterized in this manner for a unique f satisfying

$$1 - \sum_{i,j=1}^{n-1} g^{ij} D_i f D_j f > 0.$$

(This is exactly the condition that the image of F_f is spacelike.) When the image is spacelike set

$$X_i := \frac{\partial}{\partial x^i} + D_i f \frac{\partial}{\partial x^n}, \quad W := \left(1 - \sum_{i,j}^{n-1} g^{ij} D_i f D_j f\right)^{\frac{1}{2}},$$

$$\mathbf{n} := \frac{1}{W}\left(\frac{\partial}{\partial x^n} + \sum_{i,j=1}^{n-1} g^{ij} D_i f \frac{\partial}{\partial x^j}\right).$$

Then $X_1,\dots,X_{n-1}$ is a basis for the tangent space to the image of N_f and $\mathbf{n}$ is the future pointing timelike unit normal to N_f. Now a tedious calculation shows that the second fundamental form h of N_f is given by

$$h(X_i,X_j) = \frac{1}{W}\left(D_{ij} f + \Gamma_{ij}^n - V_{ij}\right)$$

where Γ_{ij}^k are the Christoffel symbols and V_{ij} are defined by

$$V_{ij} := \sum_{k=1}^{n-1} \left(\Gamma_{ij}^k D_k f + \Gamma_{in}^k D_k f D_j f + \Gamma_{jn}^k D_k f D_i f\right). \tag{3.3}$$

Solving for the Hessian of f in terms of the second fundamental form of N_f gives

$$D_{ij} f = W h(X_i,X_j) - \Gamma_{ij}^n + V_{ij}. \tag{3.4}$$

The induced metric on N_f has its components in the coordinate system $x^1,\dots,x^{n-1}$ given by

$$G_{ij} = g(X_i,X_j) = g_{ij} - D_i f D_j f.$$

Let $[G^{ij}] = [G_{ij}]^{-1}$. Then the mean curvature of N_f is

$$\begin{aligned} H &= \frac{1}{n-1} \operatorname{trace}_G h = \frac{1}{n-1} \sum_{i,j=1}^{n-1} G^{ij} h(X_i,X_j) \\ &= \frac{1}{(n-1)W} \sum_{i,j=1}^{n-1} G^{ij}\left(D_{ij} f + \Gamma_{ij}^n - V_{ij}\right), \end{aligned}$$

where $x := (x^1, \dots, x^{n-1})$,

$$[G^{ij}(x, f, Df)] = [g_{ij}(x, f) - D_i f(x) D_j f(x)]^{-1}$$

and $V_{ij}(x, f, Df)$ is given by (3.3). Now we can write

$$H = \sum_{i,j=1}^{n-1} a^{ij}(x, f, Df) D_{ij} f + b(x, f, Df).$$

where a^{ij} and b are given by

$$a^{ij}(x, f, Df) := \frac{1}{(n-1)W} G^{ij}(x, f, Df),$$
$$b(x, f, Df) := \frac{1}{(n-1)W} \sum_{i,j=1}^{n} G^{ij} \left(\Gamma^n_{ij} - V_{ij}\right)$$

Therefore if $\mathcal{H}[f]$ is the mean curvature of N_f then the operator $f \mapsto \mathcal{H}[f]$ is quasi-linear.

3.2. **Proof of Theorem 3.6.** Given an indexed set of functions $\{f_\alpha\}$ with domains Ω_α so that the hypersurfaces N_{f_α} are all spacelike, consider the fields of future pointing unit timelike normals to these hypersurfaces:

$$\mathbf{n}_\alpha = \frac{\partial}{\partial x^n} + \frac{1}{W_\alpha} \sum_{i,j=1}^{n-1} g^{ij} D_i f_\alpha \frac{\partial}{\partial x^j}, \quad \text{where} \quad W_\alpha := \left(1 - \sum_{i,j}^{n-1} g^{ij} D_i f_\alpha D_j f_\alpha\right)^{\frac{1}{2}}$$

The following will be used to guarantee the mean curvature operator on the support functions to N_0 and N_1 is uniformly elliptic in the sense of Definition 2.2. The proof is left to the reader.

Lemma 3.8. *Let $\cup_\alpha \Omega_\alpha \subset K$ where K is compact. Then there is a compact subset $\widehat{K}$ of the tangent bundle $T(M)$ that contains the set $\cup_\alpha \{\mathbf{n}_\alpha(x) : x \in \Omega_\alpha\}$ if and only if there is a $\rho_0 > 0$ so that for all α the lower bound $W_\alpha(x) \ge \rho_0$ holds for $x \in \Omega_\alpha$. Moreover if this lower bound holds and $0 < \rho < \rho_0$, there is a bound $|f_\alpha| < B$, and if $U = U_{\rho,B,K} \subset \mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^{n-1}$ is defined by*

$$(3.5) \quad U = U_{\rho,B,K} := \Bigg\{ (x^1, \dots, x^{n-1}, r, p_1, \dots, p_{n-1}) = (x, r, p) : \\ x \in K, |r| < B, \sum_{i,j=1}^{n-1} g^{ij}(x, r) p_i p_j < 1 - \rho^2 \Bigg\},$$

then for any $x \in K$ the fiber $U_x = \{((r, p) : (x, r, p) \in U\}$ is convex and for all α, the functions f_α are U–admissible over Ω_α. Finally, the mean curvature operator $\mathcal{H}$ is uniformly elliptic on U. $\square$

Let N_0 and N_1 be as in the statement of the geometric form of the maximum principle, Theorem 3.6. Choose a coordinate system $(x^1, \dots, x^n)$ centered at $q_0 \in N_0 \cap N_1$ that puts the metric of (M, g) in the form (3.2).

By choosing the coordinate neighborhood sufficiently small we insure that, within this neighborhood, N_0 and N_1 are acausal and edgeless, and N_0 is in the causal future of N_1.

Then there is an open connected set $\Omega \subseteq \mathbf{R}^{n-1}$ and continuous functions $u_0, u_1 : \Omega \to \mathbf{R}$ so that $N_0 = N_{u_0}$ and $N_1 = N_{u_1}$ as above (that is $N_i = \{(x^1, \dots, x^{n-1}, u_i(x^1, \dots, x^{n-1})) : (x^1, \dots, x^{n-1}) \in \Omega\}$). Then $u_1 \le u_0$ as N_0 is in the causal future of N_1. Define $U := U_{\rho_2, B, \Omega}$ by equation (3.5), where ρ_2 is to be chosen shortly. Then, using that N_1 has mean curvature ≥ 0 in the sense of support hypersurfaces with one–sided Hessian bounds, the discussion above (cf. Lemma 3.8 and by possibly making Ω a little smaller) there is a choice of $\rho_2 \in (0,1)$ and $B > 0$ so that every C^2 past support hypersurface P to N_1 has its unit normal at the point of contact with N_1 in the set $U = U_{\rho_2, B, \Omega}$ and the closure of U in $T(M)$ is compact. Moreover, from the calculations of §3.1, we see that any C^2 past support hypersurface $P_{q,\varepsilon}$ will be of the form $P_{q,\varepsilon} = N_{\psi_{q,\varepsilon}}$ where $\psi_{q,\varepsilon}$ is a C^2 U–admissible lower support function to u_1 at q that satisfies $\mathcal{H}[\psi_{q,\varepsilon}](q) \ge H_0 - \varepsilon$ ($\mathcal{H}$ is the mean curvature operator) and which has a lower bound on its Hessian. Thus in Ω we see $\mathcal{H}[u_1] \ge H_0$ in the sense of support functions with a one–sided Hessian bound. Thus $\mathcal{H}$ and u_1 satisfy the hypothesis of the analytic maximum principle.

Likewise if S is a C^2 future support hypersurface for N_0 then there is a C^2 upper support function φ_0 for u_0 such that locally $S = N_{\varphi_0}$. The problem, as noted in Remark 3.4, is that φ_0 need not be U–admissible. Now use the notation of the standard setup (with n replaced by $n-1$) except that we do not know that the upper support function φ_0 to u_0 at x_* is U–admissible. As the set U is open and the function u_0 is continuous there is an $r_2 > 0$ and $\delta_2 > 0$ so that

(3.6)
$$\|x_* - x_1\| \le r_2, \text{ and } \|p - D\varphi_1(x_*)\| \le \delta_2 \quad \text{implies} \quad (x_*, u_0(x_*), p) \in U$$

We now make a few modifications in §2.4, where the proof of the analytic maximum principle Theorem 2.4 was given. Change the definition of r_1 to

$$r_1 = \min \left\{ r_0, r_2, \ \frac{1}{4\left(n^2 C_{\mathbf{E}}(C_{\mathbf{H}}+1)\right)\left(\overline{\alpha} r_0^{-(\overline{\alpha}+2)}\right)} \right\}$$

where $\overline{\alpha}$ is given by

$$\overline{\alpha} := -2 + C_{\mathbf{E}}\left(1 + (n-1)C_{\mathbf{E}} + (n-1)^3 C_{\mathbf{E}}(C_{\mathbf{H}}+1)\right).$$

(This is equation (2.23) with n replaced by $n-1$.) Then $\overline{\alpha}$ only depends on $C_{\mathbf{E}}$, $C_{\mathbf{S}}$, r_0, and $n-1$.

If $f = \varphi_1 - \varphi_0 + \delta w$ has a local maximum at x_* then by Lemma 2.8

$$\|D\varphi_1(x_*) - D\varphi_0(x_*)\| \le \delta \alpha r_0^{-(\alpha+2)}.$$

Therefore define

$$\delta_3 = \delta_2 \left(\overline{\alpha} r_0^{-(\overline{\alpha}+2)}\right)^{-1}$$

so that $\delta_3\left(\overline{\alpha}r_0^{-(\overline{\alpha}+2)}\right) = \delta_2$. Now in the proof of the analytic maximum principle change the definition of δ to

$$\delta = \min\{\delta_1, \delta_3, \overline{\delta}(\overline{\alpha})\}$$

where $\overline{\delta}(\alpha)$ is still defined by equation (2.13) and δ_1 has the same definition as in §2.4. Then using these definitions of r_1 and δ in the proof along with the fact that the bound just given on $\|D\varphi_1(x_*)-D\varphi_0(x_*)\|$, implication (3.6) and the definition of δ_3 imply that, as $f = \varphi_1-\varphi_0+\delta w$ has a local maximum at x_*, φ_0 will be U–admissible. The rest of the proof proceeds exactly as before to show that $u_0 = u_1$ in Ω. This implies that N_0 and N_1 agree in a neighborhood $\mathcal{O}$ of q_0.

Finally note that as the metric g is smooth the functions a^{ij} and b in the definition of the mean curvature operator $\mathcal{H}$ are C^∞. Thus the regularity part of Theorem 2.4 implies that $N_1\cap\mathcal{O} = N_2\cap\mathcal{O}$ is a smooth hypersurface. This completes the proof of Theorem 3.6.

3.3. **A Geometric Maximum Principle for Riemannian Manifolds.** We now fix our sign conventions on the imbedding invariants of smooth hypersurfaces in a Riemannian manifold (M, g). It will be convenient to assume that our hypersurfaces are the boundaries of open sets. As this is always true locally it is not a restriction. As in the Lorentzian case we denote the metric connection by ∇. Let $D \subset M$ be a connected open set and let $N \subset \partial D$ be part of the boundary that is a C^2 hypersurface (we do not want to assume that all of ∂D is smooth). Let $\mathbf{n}$ be the outward pointing unit normal along N. Then the second fundamental form of N is the symmetric bilinear form defined on the tangent spaces to N by $h^N(X,Y) = \langle\nabla_X Y, \mathbf{n}\rangle$. The mean curvature of N is then $H^N := \frac{1}{n-1}\operatorname{trace}_{g\big|_N} h^N = \frac{1}{n-1}\sum_{i=1}^{n-1} h^N(e_i, e_i)$ where $e_1, \dots, e_{n-1}$ is a local orthonormal frame for $T(N)$. This is the sign convention so that for the boundary S^{n-1} of the unit ball B^n in $\mathbf{R}^n$ the second fundamental form $h^N = -g\big|_{S^n}$ is negative definite and the mean curvature is $H^{S^{n-1}} = -1$.

Definition 3.9. Let U be an open set in the Riemannian manifold (M, g). Then

1. ∂U has mean curvature $\geq H_0$ **in the sense of contact hypersurfaces** iff for all $q \in \partial U$ and $\varepsilon > 0$ there there is an open set D of M with $\overline{D} \subseteq \overline{U}$, $q \in \partial D$, and the part of ∂D near q is a C^2 hypersurface of M and at the point q, $H_q^{\partial D} \geq H_0 - \varepsilon$.
2. ∂U has mean curvature $\geq H_0$ **in the sense of contact hypersurfaces with a one sided Hessian bound** iff for all compact $K \subset \partial U$ there is constant $C_K \geq 0$ so that for all $q \in K$ and $\varepsilon > 0$ there is a an open set D of M with $\overline{D} \subseteq \overline{U}$, $q \in \partial D$, the part of ∂D near q is a C^2 hypersurface of M and at the point q, $H_q^{\partial D} \geq H_0 - \varepsilon$ and also $h_q^{\partial D} \geq -C_K g\big|_{\partial D}$. □

The proof of the Lorentzian version of the geometric maximum principle can easily be adapted to prove the following Riemannian version.

Theorem 3.10 (Geometric Maximum Principle for Riemannian Manifolds)**.** *Let (M, g) be a Riemannian manifold, $U_0, U_1 \subset M$ open sets, and let H_0 be a constant. Assume that*

1. $U_0 \cap U_1 = \emptyset$,
2. ∂U_0 *has mean curvature* $\geq -H_0$ *in the sense of contact hypersurfaces,*
3. ∂U_1 *has mean curvature* $\geq H_0$ *in the sense of contact hypersurfaces with a one sided Hessian bound, and*
4. *there is a point* $p \in \overline{U}_0 \cap \overline{U}_1$ *and a neighborhood* $\mathcal{N}$ *of* p *that has coordinates* $x^1, \ldots, x^n$ *centered at* p *so that for some* $r > 0$ *the image of these coordinates is the box* $\{(x^1, \ldots, x^n) : |x^i| < 0\}$ *and there are Lipschitz continuous functions* $u_0, u_1 : \{(x^1, \ldots, x^{n-1}) : |x^i| < r\} \to (-r, r)$ *so that* $U_0 \cap \mathcal{N}$ *and* $U_1 \cap \mathcal{N}$ *are given by*

$$U_0 \cap \mathcal{N} = \{(x^1, \ldots, x^n) : x^n > u_0(x^1, \ldots, x^{n-1})\},$$
$$U_1 \cap \mathcal{N} = \{(x^1, \ldots, x^n) : x^n < u_1(x^1, \ldots, x^{n-1})\}.$$

(This implies $u_1 \leq u_0$ *and* $u_1(0, \ldots, 0) = u_0(0, \ldots, 0)$*).*

Then $u_0 \equiv u_1$ *and* u_0 *is a smooth function. Therefore* $\partial U_0 \cap \mathcal{N} = \partial U_1 \cap \mathcal{N}$ *is a smooth embedded hypersurface with constant mean curvature* H_0 *(with respect to the outward normal to* U_1*).* □

In proving this, note that the Lipschitz conditions on u_0 and u_1 makes the mean curvature operator uniformly elliptic in the sense of Definition 2.2 when applied to upper and lower support functions to u_0 and u_1.

Remark 3.11. This is a special case of a more general result which we give in another paper [21]. There the conditions on the boundaries ∂U_i are relaxed to assuming only that at each point p there is an open ball $B \subset U_i$ that has p in its closure and that the radii of these balls is locally bounded from below (this allows sets where the boundaries do not have to be topological manifolds). Most of the work in proving the more general maximum principle involves proving a structure theorem for the boundaries of open sets that satisfy a "locally uniform inner sphere condition" as above, but once this is done a main step in the proof is exactly the version of the maximum principle given in Theorem 3.10. □

4. Applications to Warped Product Splitting Theorems

4.1. **Statement of Results.** The Lorentzian geometric maximum principle, Theorem 3.6, provides an especially natural and conceptually transparent proof of the Lorentzian splitting theorem. Recall, the Lorentzian splitting theorem asserts that if a globally hyperbolic or timelike geodesically complete spacetime M obeys the strong energy condition, $\mathrm{Ric}(T, T) \geq 0$ for all timelike vectors T, and contains a complete timelike line $\gamma : (-\infty, \infty) \to M$ then M splits isometrically into a Lorentzian product. The proof makes

use of Lorentzian Busemann functions (see e.g., [7] for a nice introduction). Let b^+ be the Busemann function associated to the ray $\gamma|_{[0,\infty)}$, and let b^- be the Busemann function associated to the ray $-\gamma|_{[0,\infty)}$. The regularity theory of Lorentzian Busemann functions (cf. [16] and [7, §14.1-§14.3]) guarantees that $b^\pm$ are continuous on a neighborhood $\mathcal{O}$ of $\gamma(0)$ and that $N^\pm = \{b^\pm = 0\}$ are C^0 spacelike hypersurfaces in the sense of Definition 3.1 in $\mathcal{O}$. $N^\pm$ both pass through $\gamma(0)$, and, by the reverse triangle inequality, N^- is locally to the future of N^+ near $\gamma(0)$. The curvature assumption implies that N^- has mean curvature ≤ 0 in the sense of support hypersurfaces and N^+ has mean curvature ≥ 0 in the sense of support hypersurfaces with one–sided Hessian bounds. Thus, Theorem 3.6 implies that N^+ and N^- agree and are smooth near $\gamma(0)$. It follows that $b^+ = b^- = 0$ along a smooth maximal (i.e., mean curvature zero) spacelike hypersurface N. The original proof of this given in [15] is less direct and less elementary, as it makes use of a deep existence result for maximal hypersurfaces due to Bartnik [6]. It is then straight-forward to show that the normal exponential map along N gives a splitting of a tubular neighborhood of γ. This local splitting can then be globalized.

The following theorem extends the Lorentzian splitting theorem to spacetimes which satisfy the strong energy condition with positive cosmological constant.

Theorem 4.1 (Warped product splitting theorem)**.** *Let (M, g) be a connected globally hyperbolic spacetime which satisfies $\mathrm{Ric}(T, T) \geq n - 1$ for any timelike unit vector T. Assume there is a timelike arclength parameterized geodesic segment $\gamma : (-\pi/2, \pi/2) \to M$ which maximizes the distance between any two of its points. Then there is a complete Riemannian manifold (N, g_N) such that (M, g) is isometric to a warped product $(-\pi/2, \pi/2) \times N$ with metric*

$$-dt^2 + \cos(t)^2 g_N. \tag{4.1}$$

Remark 4.2. A calculation shows that a spacetime with metric of the form (4.1) satisfies the assumptions of Theorem 4.1 if the sectional curvature of (M, g) is bounded from below by -1. □

The following notion which is weaker than global hyperbolicity was used by Harris in [18].

Definition 4.3. Let (M, g) be a Lorentzian manifold and let $q > 0$. Then (M, g) is globally hyperbolic of order q if and only if M is strongly causal and $x \ll y$, $d(x, y) < \pi/q$ implies that $C(x, y)$ is compact, where $C(x, y)$ is the set of causal curves connecting x and y. □

To state our next result we need some notation. Let $(\mathbf{H}^{n-1}, g_{\mathbf{H}})$ be the standard Riemannian hyperbolic space with sectional curvature $\equiv -1$ and let $(\mathbf{S}^n_1(-1), g_{\mathbf{S}})$ be the Lorentzian manifold $\mathbf{S}^n_1(-1) := \mathbf{H}^{n-1} \times (\pi/2, \pi/2)$ with the metric $g_{\mathbf{S}} := \cos^2(t) g_{\mathbf{H}} - dt^2$. This is a Lorentzian manifold

(the subscript in $\mathbf{S}_1^n(-1)$ is to indicate the metric has index one, i.e. it is Lorentzian) which has sectional curvature $\equiv -1$. Recall that the unique simply connected geodesically complete Lorentzian manifold with sectional curvature $\equiv -1$, which we will denote by $\mathbf{R}_1^n(-1)$, is the universal anti-de Sitter space (cf. [7, p. 183]). While $(\mathbf{S}_1^n(-1), g_{\mathbf{S}})$ has sectional curvature $\equiv -1$ it is not geodesically complete and so is not isometric to $\mathbf{R}_1^n(-1)$ but is isometric to an open subset of $\mathbf{R}_1^n(-1)$. If $x \in \mathbf{H}^{n-1}$ then the curve $\gamma_x : (\pi/2, \pi/2) \to \mathbf{S}_1^n(-1)$ given by $\gamma_x(t) = (x, t)$ is a timelike unit speed geodesic and moreover $(\mathbf{S}_1^n(-1), g_{\mathbf{S}})$ is globally hyperbolic and if x is any point of $\mathbf{S}_1^n(-1)$ then there are points p and q on γ_x so that $p \ll x \ll q$. Using these facts it is not hard to show that given any unit speed timelike geodesic $\gamma : (\pi/2, \pi/2) \to \mathbf{R}_1^n(-1)$ of length π, the set $\{x \in \mathbf{R}_1^n(-1) : \text{there are } p \text{ and } q \text{ on } \gamma \text{ so that } p \ll x \ll q\}$ is isometric to $(\mathbf{S}_1^n(-1), g_{\mathbf{S}})$.

Corollary 4.4 (Lorentzian maximal diameter theorem)**.** *Let (M, g) be a connected Lorentzian manifold which is globally hyperbolic of order 1 and assume that* $\operatorname{Ric}(T, T) \geq (n-1)$ *for any timelike unit vector T. If M contains a timelike geodesic segment $\gamma : [-\pi/2, \pi/2] \to M$ of length π connecting x and y, then $D = \{z : x \ll z \ll y\}$ is isometric to $(\mathbf{S}_1^n(-1), g_{\mathbf{S}})$. Moreover, if M contains a timelike geodesic $\gamma : (-\infty, \infty) \to M$ such that each segment $\gamma\big|_{[t,t+\pi]}$ is maximizing, then (M, g) is isometric to the universal anti-de Sitter space $\mathbf{R}_1^n(-1)$.*

A very closely related result is given by Eschenburg [12, Cor. 2 p. 66].

Our next result makes use of the notion of the cosmological time function (Definition 4.5) which was introduced and studied in [4] as a canonical choice of a global time function for cosmological spacetimes.

Definition 4.5. Let (M, g) be a spacetime. Then the **cosmological time function** $\tau : M \to (0, \infty]$ is defined by

$$\tau(q) = \sup\{d(p, q) : p \ll q\}.$$

where d is the Lorentzian distance function. □

If (M, g) is spacetime of dimension ≥ 4 let W_g be the Weyl conformal tensor of (M, g) written as a $(0, 4)$ tensor. Then the squared norm of W with respect to the metric g is

$$\|W_g\|_g^2 = \sum g^{AA'} g^{BB'} g^{CC'} g^{DD'} W_{ABCD} W_{A'B'C'D'}.$$

Under a conformal change of metric $\tilde{g} = \lambda^2 g$ this transforms as

$$W_{\tilde{g}} = \lambda^2 W_g, \quad \|W_{\tilde{g}}\|_{\tilde{g}}^2 = \lambda^{-4} \|W_g\|_g^2. \tag{4.2}$$

Because the metric g is not positive definite it is possible that $\|W_g\|_g^2 = 0$ at a point without W_g being zero at the point and in general $\|W_g\|_g^2$ can be negative.

Theorem 4.6. *Let (M, g) be a globally hyperbolic spacetime of dimension $n \geq 4$. Assume*

1. $\mathrm{Ric}(T, T) \geq (n-1)$ *on all timelike unit vectors T.*
2. *There is a geodesic $\gamma : (-\pi/2, \pi/2) \to M$ that maximizes the distance between any two of its points.*
3. *The Weyl conformal tensor W_g of g and the cosmological time function $\tau : M \to (0, \infty]$ satisfy*
$$\lim_{\tau(q) \searrow 0} \tau(q)^4 \|W_g\|_g^2 = 0.$$

Then there is a complete Riemannian manifold (N, g_N) of constant sectional curvature such that (M, g) is isometric to a warped product $(-\pi/2, \pi/2) \times N$ with metric
$$-dt^2 + \cos(t)^2 g_N.$$

Remark 4.7. This result is loosely related to the Weyl curvature hypothesis of R. Penrose, see [24]. Theorem 4.6 can be restated as saying that if $\mathrm{Ric}(T, T) \geq n - 1$ on timelike unit vectors T, there is a line of length π and the Weyl conformal tensor W_g has order $o(\tau^{-2})$ (so that the squared norm $\|W_g\|_g^2$ has order $o(\tau^{-4})$ then (M, g) is "locally spatially isotropic". $\square$

4.2. **Proofs.**

4.2.1. *Proof of Theorem 4.1.* Let (M, g) be an n dimensional globally hyperbolic spacetime so that $\mathrm{Ric}(T, T) \geq (n-1)$ for all timelike unit vectors T. Call a unit speed timelike geodesic $\gamma : (-\pi/2, \pi/2) \to M$ that maximizes the distance between an two of its points a **line** in M. By hypothesis M has at least one line. For any curve (or any subset of M) $c : (a, b) \to M$ let $I^+(c) = \{x : \text{there is a } p \text{ on } c \text{ with } p \ll x\}$, $I^-(c) = \{x : \text{there is a } q \text{ on } c \text{ with } x \ll q\}$ and $I(c) = I^+(c) \cap I^-(c)$.

Definition 4.8. Let $\gamma : (-\pi/2, \pi/2) \to M$ be a line in M, and let $s \in (-\pi/2, \pi/2)$. For $p \ll \gamma(s)$, let α_s be a maximal geodesic connecting p and $\gamma(s)$. If there is a sequence $\{s_k\}_{k=1}^{\infty} \subset (-\pi/2, \pi/2)$ and a timelike unit vector v such that $s_k \nearrow \pi/2$, $p \ll \gamma(s_k)$ and $\dot{\alpha}_{s_k}(0) \to v \in T(M)_p$ then the maximal geodesic starting at p in the direction v is called an **asymptote** to γ at p. $\square$

If $(-\pi/2, \pi/2) \to M$ is a line in M then for each $r \in (-\pi/2, \pi/2)$ define
$$b_r(x) := d(\gamma(0), \gamma(r)) - d(x, \gamma(r)) = r - d(x, \gamma(r))$$
where d is the Lorentzian distance function. The **Busemann function** of γ is then defined by
$$b(x) := \lim_{r \nearrow \pi/2} b_r(x). \tag{4.3}$$

For fixed $x \in M$ the reverse triangle inequality (which is $d(p, q) + d(q, z) \leq d(p, z)$ if $p \ll q \ll z$) implies the function $r \mapsto b_r(x)$ is monotone decreasing

thus $b(x)$ exists for all $x \in M$. Most of the following is contained in the papers of Eschenburg [11] and Galloway [15].

Proposition 4.9. *Let* (M, g) *be as in the statement of Theorem 4.1 and* $\gamma : (-\pi/2, \pi/2) \to M$ *be a line in* M*. Then there is a neighborhood* U *of* $\gamma(0)$ *(a* **nice neighborhood***) so that*

1. *The Busemann function of* γ *is continuous on* U *and satisfies the reverse Lipschitz inequality*

$$b(q) \geq b(p) + d(p, q) \quad \textit{for} \quad p \ll q. \tag{4.4}$$

2. *For each* $x \in U$ *there is an asymptote* α_x *at* x *and it satisfies*

$$b(\alpha_x(t)) = b(x) + t \quad \textit{for} \quad 0 \leq t < \pi/2 - b(x). \tag{4.5}$$

   Moreover α_x *is a ray in the sense that it is future inextendible and maximizes the distance between any two of its points.*

3. *The set of initial vectors* $\{\alpha'_x(0) : x \in U\}$ *is contained in a compact subset of* $T(M)$*.*
4. *The zero set* $N^+ := \{x \in U : b(x) = 0\}$ *of* b *in* U *is a* C^0 *spacelike hypersurface (Definition 3.1) in* U *in and the mean curvature of* B^+ *satisfies* $H \geq 0$ *in the sense of support hypersurfaces with one-sided Hessian bounds.*

Proof. Parts (1), (2), and (3) are contained in [11] and [15].

It remains to prove the bound on the mean curvature of N^+. Let $x \in N^+$ and let $\eta = \alpha'_x(0)$. Then set $r_0 = \pi/4$, $\delta = \pi/8$ and for $r \in [\pi/4, \pi/2)$ let $S_{\eta,r}$ be defined by equation (3.1). Then as $\exp(t\eta) = \alpha_x(t)$ is a ray this implies, using the notation of Proposition 3.5, that $\gamma_\eta(t) := \exp(t\eta) = \alpha_x(t)$ maximizes distance on $[0, r_0 + \delta]$. Also as α_x is a ray the segment $\gamma_\eta\big|_{[0,r]}$ maximizes the distance between $x = \alpha_x(0)$ and $\exp(r\eta) = \alpha_x(r)$ this segment contains no cut points. This implies the past geodesic sphere $S_{\eta,r}$ is smooth near the point x, and by Proposition 3.5 there will be a uniform lower bound on the second fundamental forms of $h_x^{S_{\eta,r}}$ for $r \geq r_0 = \pi/2$ as x varies over $U \cap N^+$. By a standard comparison theorem the mean curvature of $S_{\eta,r}$ at x satisfies $H_x^{S_{\eta,r}} \geq -\cot(r)$. Thus for any $\varepsilon > 0$ if we choose $r \in [\pi/4, \pi/2)$ so that $\cot(r) < \varepsilon$ then $S_{\eta,r}$ will be past support hyperplane for N^+ at x with mean curvature at $x > -\varepsilon$. (That $S_{\eta,r}$ is in the past N^+ follows from the reverse Lipschitz inequality.) This completes the proof. □

We first prove a local version of Theorem 4.1. Given the line γ, let $b^+ : M \to \mathbf{R}$ be defined by equation (4.3) above. For $r \in (-\pi/2, 0]$ let

$$b_r^-(x) := d(\gamma(r), \gamma(0)) - d(\gamma(r), x) = -r - d(\gamma(r), x)$$

and

$$b^- : (x) = \lim_{r \searrow -\pi/2} b_r^+(x).$$

This is just the Busemann function for γ if the time orientation of M is reversed and so the obvious variant of the last proposition shows that there

is a neighborhood U of $\gamma(0)$ so that if $N^\pm := \{x \in U : b^\pm(x) = 0\}$, then both N^+ and N^- are C^0 spacelike hypersurface in U, the mean curvature of N^+ is ≥ 0 in the sense of support hypersurfaces with one–sided Hessian bounds and the mean curvature of N^- is ≤ 0 in the sense of support hypersurfaces.

By the reverse triangle inequality

$$\begin{aligned} b^+(x) + b^-(x) &= \lim_{r \nearrow \pi/2} (b^+_r(x) + b^-_{-r}(x)) \\ &= \lim_{r \nearrow \pi/2} (d(\gamma(0), \gamma(r)) + d(\gamma(-r), \gamma(0)) \\ &\qquad\qquad - d(x, \gamma(r)) - d(\gamma(-r), x)) \\ &= \lim_{r \nearrow \pi/2} (d(\gamma(-r), \gamma(r)) - d(x, \gamma(r)) - d(\gamma(-r), x)) \\ &\geq 0. \end{aligned} \tag{4.6}$$

Thus if $x \in N^-$, so that $b^-(x) = 0$, then $b^+(x) = b^+(x) + b^-(x) \geq 0$. But $\{x : b^+(x) \geq 0\}$ is in the causal future of the set $\{x : b^+(x) = 0\}$. This implies that N^- is locally to the future of N^+ near $\gamma(0)$. Therefore the hypersurfaces N^+ and N^- satisfy the hypothesis of the geometric maximum principle 3.6. Thus, by choosing U smaller, if necessary, $N^+ = N^-$ and this is a smooth hypersurface with mean curvature $H \equiv 0$.

For any point $x \in N^+$ any asymptote α_x to γ must be orthogonal to N^+. This is because N^+ is smooth and the support hypersurfaces we used to show N^+ has mean curvature ≥ 0 in the sense of support hypersurfaces where orthogonal to α_x at x and as N^+ is smooth it must have the same tangent planes as any of its support hypersurfaces at a point of contact. As any asymptote to γ from x must be orthogonal to N^+ this also shows that at points $x \in N^+$ that the asymptote to γ from x is unique and is given by $\alpha_x(t) = \exp(t\mathbf{n}(x))$ for $0 \leq t \leq \pi/2$ where $\mathbf{n}(x)$ is the future pointing unit normal to N^+ at x. From the inequality (4.4) and the identity (4.5) we see that the ray α_x maximizes the distance between any of its points $\alpha_x(t) = \exp(t\mathbf{n}(x))$ and N^+ for $t \in [0, \pi/2)$. Thus the hypersurface N^+ has no focal points along α_x. But N^+ has mean curvature $= 0$ and $\mathrm{Ric}(\alpha_x'(t), \alpha_x'(t)) \geq n - 1$. A standard part of the comparison theory implies that N has a focal point along α_x at a distance $\leq \pi/2$ and if equality holds then the second fundamental form of N^+ vanishes at x and for each $t \in [0, \pi/2)$ the curvature tensor of (M, g) satisfies $R_{\alpha_x(t)}(X, \alpha_x'(t))\alpha_x'(t) = X$ for all vectors X orthogonal to $\alpha_x'(t)$. (Cf. [19] for the Riemannian version which is easily adapted to the Lorentzian setting.) Note that the second fundamental form of N^+ vanishes at all points and thus N^+ is totally geodesic.

Proposition 4.10. *Define a map $\Phi : N^+ \times (-\pi/2, \pi/2) \to M$ by $\Phi(x, t) = \exp(t\mathbf{n}(x))$ and let $V := \{\exp(t\mathbf{n}(x)) : x \in N^+, t \in (-\pi/2, \pi/2)\}$ be the image of Φ. Then Φ is injective and V has a warped product metric of the required type. That is*

$$\Phi^* g = -dt^2 + \cos(t)^2 g\big|_{N^+}. \tag{4.7}$$

Also in V the Busemann functions of γ are given by $b^+(\Phi(x,t)) = t$ and $b^-(\Phi(x,t)) = -t$. Thus b^+ and b^- are smooth in a neighborhood of γ and moreover γ is an integral curve of the gradient field $-\nabla b^+ = \partial/\partial t$ and the level sets of $b^+ = t$ are smooth hypersurfaces in V and so the distribution $(\nabla b^+)^\perp$ is integrable.

Proof. Consider the restriction $\Phi\big|_{N_+ \times [0,\pi/2)} : N_+ \times [0,\pi/2) \to M$. Then a direct calculation using the form of the curvature along the geodesics α_x shows that $\Phi^* g = -dt^2 + \cos(t)^2 g_{N^+}$. To see this map is injective assume $\Phi(x_1, s) = \Phi(x_2, t)$ with $s, t > 0$. Then as the second argument of Φ is the distance from N_+ we have $s = t$. Now assume that $x_1 \neq x_2$ but $\Phi(x_1, s) = \Phi(x_2, s)$. Then choose $s_1 \in (s, \pi/2)$ and let β be the broken geodesic form x_1 to $\alpha_{x_2}(s_1)$ formed by following α_{x_1} from x_1 to $\alpha_{x_1}(s) = \alpha_{x_2}(s)$ and then following α_{x_2} from $\alpha_{x_1}(s) = \alpha_{x_2}(s)$ to $\alpha_{x_2}(s_1)$. The length of β is s_1, but the corner of β can be smoothed to find a curve β_1 near β that has length greater than s_1. But this contradicts that α_{x_2} realizes the distance between N_+ and its point $\alpha_{x_2}(s_1)$ and so $\Phi\big|_{N_+ \times [0,\pi/2)}$ is injective. Therefore the set $V^+ := \{\exp(t\mathbf{n}(x)) : x \in N^+, 0 \leq t < \pi/2\}$ has a warped product metric of the required type. Similar considerations working with N^-, b^-, and the past pointing unit normal to N^- shows that the set $V^- := \{\exp(t\mathbf{n}(x)) : x \in N^+, -\pi/2 < t \leq 0\}$ also has a warped product metric of the required type. But as $N^+ = N^-$ the two sets V^+ and V^- piece together to give that $V := \{\exp(t\mathbf{n}(x)) : x \in N^+, -\pi/2 < t \leq \pi/2\}$ has a warped product product metric as claimed. The facts about the Busemann functions are now clear. □

All that remains is to show that local warped product splitting implies global warped product splitting.

Definition 4.11. A **strip** is a totally geodesic immersion f of

$$\big((-\pi/2, \pi/2) \times I, -dt^2 + \cos^2(t) ds^2\big)$$

into M for some interval I so that $f\big|_{(-\pi/2,\pi/2)\times\{s\}}$ is a timelike line for each $s \in I$. We will denote by S the space $(-\pi/2, \pi/2) \times \mathbf{R}$ and by g_S the metric $-dt^2 + \cos^2(t) ds^2$. □

We will make use of the following elementary facts.

Lemma 4.12. *Consider (S, g_S) as just defined and let $d_q(x) = d(q, x)$ for q, x causally related. Then for $0 < a < \pi$*

$$\{d_{(\pi/2,0)} = a\} = \{\pi/2 - a\} \times \mathbf{R}.$$

Similarly, let $\gamma_1(t) = (t, s_1)$ and $\gamma_2(t) = (t, s_2)$ be two lines. Then

$$\lim_{t \to \pi/2} d(\gamma_1(t), \gamma_2(\tau)) = \pi/2 - \tau. \quad \square$$

The following is analogous to [11, Prop. 7.1], see also the remarks on p. 384 of [15].

Proposition 4.13. *Let $\gamma : (-\pi/2, \pi/2) \to M$ be a timelike line and let $\sigma : [0, \ell] \to M$ be a geodesic with $\sigma(0) = \gamma(0)$. Then there is a strip containing γ and σ.*

Proof. Suppose there is a strip $f : (-\pi/2, \pi/2) \times [0, a) \to M$ containing γ and $\sigma\big|_{[0,v)}$ so that for all t, $\gamma(t) = f(t, 0)$, where v is the defined as

$$v = \sup\{u : \sigma\Big|_{[0,u)} \subset f((-\pi/2, \pi/2) \times [0, a))\}.$$

Now consider the geodesic λ in the space $U := (-\pi/2, \pi/2) \times \mathbf{R}$ with the metric (4.7) with $f_*\dot\lambda(0) = \dot\sigma(0)$. Define $X_0 \in T(N)_{(0,0)}$ by parallel translating $\partial/\partial t$ from $\lambda(v)$ to $\lambda(0)$ along λ. Then define a vector $X \in T(M)_{\sigma(v)}$ by parallel translating f_*X_0 along σ from $\sigma(0)$ to $\sigma(v)$. By construction, X is a timelike unit vector.

Let (t_0, s_0) be the coordinates of $\lambda(v)$ and let $\gamma_v : (a, b) \to M$ be the inextendible arclength parameterized timelike geodesic such that $\dot\gamma_v(t_0) = X$.

We claim that $(a, b) = (-\pi/2, \pi/2)$. Assume that this is not the case, for example $b < \pi/2$. Then, $\gamma_v(t)$ for $t \in [t_0, b)$ are limits of points of the form

$$f(t, s),\ (t, s) \in [t_0, b) \times [0, s_0).$$

Note that $((-\pi/2, \pi/2) \times I, -dt^2 + \cos^2(t)ds^2)$ is conformal to $(\mathbf{R} \times I, -dt'^2 + ds^2)$ (by the change of coordinates $t' = \ln(\sec(t) + \tan(t))$ for any interval I. Since the causal structure is invariant under conformal changes, we see that we may choose $\tau < \pi/2$ sufficiently large, so that $f([0, b) \times [0, s_0)) \subset J^-(\gamma(\tau))$. Therefore we find that

$$\gamma_v\Big|_{[t_0,b)} \subset J^+(\gamma_v(t_0)) \cap J^-(\gamma(\tau)).$$

But by assumption, γ_v is future inextendible so this contradicts the global hyperbolicity of M.

Therefore $b = \pi/2$ and similarly we find that $a = -\pi/2$. Further, γ_v is a limit of lines of the form $f(\cdot, s_n)$ for some sequence s_n converging to s_0 and thus γ_v is a line. Now applying the local splitting (Proposition 4.10) at γ_v we get a contradiction to the choice of v. This shows that $v = \ell$ and thus that σ is contained in the strip. □

Following [11] we say that two lines of length π are **strongly parallel** if they bound a strip and **parallel** if there is a finite sequence of lines $\gamma_1 = \sigma_1, \sigma_2, \dots, \sigma_k = \gamma_2$ such that σ_j, σ_{j+1} are strongly parallel. The following is analogous to [11, Lemma 7.2].

Lemma 4.14. *If γ_1 and γ_2 are parallel lines, then $I(\gamma_1) = I(\gamma_2)$ and the Busemann functions $b_1^\pm$ and $b_2^\pm$ of γ_1 and γ_2 agree. Thus for a line γ_1 and a point $x \in M$ there is at most one line γ_2 through x and parallel to γ_1.*

Proof. It is clear that it is sufficient to consider only the case of strongly parallel lines. It follows from the causal structure of (S, g_S) that $\gamma_1 \subset I(\gamma_2)$ and similarly that $\gamma_2 \subset I(\gamma_2)$. Therefore $I(\gamma_1) = I(\gamma_2)$.

Further, as $t \to \pi/2$, we have

$$\limsup_{t\to\pi/2} d(f(0,s), \gamma_i(t)) \geq \pi/2$$

by Lemma 4.12. Thus, for $i = 1, 2$, $b_i^+(f(s,0)) \leq 0$ and $b_i^-(f(s,0)) \leq 0$. Therefore by (4.6) we get $b_i^\pm(f(0,s)) = 0$.

This shows using (4.4) that $b_1^+(\gamma_2(s)) \geq s$. On the other hand using Lemma 4.12 and the reverse triangle inequality we have $b_1^+(\gamma_2(s)) \leq s$. Therefore we find that γ_1 is a coray of γ_2. Thus as in [11, (5)] we have $b_1^\pm \geq b_2^\pm \geq b_1^\pm$. Thus $b_1^\pm = b_2^\pm$ as claimed.

Let γ_2 and γ_3 be lines of (M, g) through x and parallel to γ_1 and let b_2^+ and b_3^+ be the Busemann functions of γ_2 and γ_3. Then we have just shown that $b_2^+ = b_3^+$. By the local version, Proposition 4.10, of the splitting we see that b_2^+ and b_3^+ are smooth functions and γ_2 and γ_3 are both integral curves for the vector field $-\nabla b_2$. Thus the uniqueness of integral curves implies $\gamma_2 = \gamma_3$. This completes the proof. □

Let γ be a fixed line and let $P_\gamma \subset M$ be the set of points which lie on a parallel line. The following Lemma is proved exactly as in [11].

Lemma 4.15 ([11, Lemma 7.3])**.** *P_γ is a connected component of M.* □

We now complete the proof of the theorem along the lines of Eschenburg [11]. The uniqueness of the line γ_q through each $q \in M$ gives a timelike vector field V on M. Let $V^\perp$ be the distribution orthogonal to V. It follows from the local version Proposition 4.10 this vector field is smooth and $V^\perp$ is integrable and the integral leaves can be represented by the level sets of the time function t.

Let N be the maximal leaf through $\gamma(0)$ and let g be the induced metric on N. Then the map

$$j : (-\pi/2, \pi/2) \times N \to M; \qquad j(t,q) \to \gamma_q(t)$$

is an isometry with respect to the metric

$$-dt^2 + \cos^2(t) g. \tag{4.8}$$

The local part of this follows from the local result Proposition 4.10 and the global statement follows from Lemma 4.15. Thus in particular $(-\pi/2, \pi/2) \times M$ with the metric (4.8) is globally hyperbolic. Therefore, by [7, Theorem 3.66 p. 103] (H, h) is a complete Riemannian manifold. This completes the proof of Theorem 4.1 □

4.2.2. *Proof of Corollary 4.4.* To prove the first statement apply Theorem 4.1 to the interior of D. This implies there is a complete Riemannian manifold (N, g_N) so that the restriction of the metric to D is a warped

product $g = -dt^2 + \cos(t)^2 g$ on $N \times (-\pi/2, \pi/2)$. We now argue that because the metric g is smooth at x that the manifold (N, g_N) must be the hyperbolic space $(\mathbf{R}_0^{n-1}(-1), g)$. It is enough to show (N, g_N) has constant sectional curvature -1. Let $N_t := N \times \{t\}$. Then the induced metric on N_t is $\cos^2(t)g$. Thus the intrinsic sectional curvatures of N_t and N_0 on corresponding 2 planes are related by $K_t = (1/\cos^2(t))K_0 g$. The distance of N_t to $x = \gamma(-\pi/2)$ is $\pi/2 + t$. Thus $N_t := \{z : x \ll z,\ d(x,z) = \pi/2 + t\}$. But in the limit as $t \searrow -\pi/2$ the set N_t is then the "geodesic sphere" of radius $r = \pi/2 + t$. As the radius goes to 0 (i.e. as $t \to -\pi/2$) the Gauss equation implies the sectional curvatures behave like $K_t = -1/r^2 + O(1/r) = -1/(\pi/2+t)^2 + O(1/(\pi/2+t))$. Therefore

$$K_t = \frac{K_0}{\cos^2(t)} = \frac{K_0}{(\pi/2+t)^2} + O\left(\frac{-1}{(\pi/2+t)}\right) = \frac{-1}{(\pi/2+t)^2} + O\left(\frac{1}{(\pi/2+t)}\right).$$

This implies $K_0 \equiv -1$.

The proof of the second statement can be carried out along the lines of [18, §2.2]. $\square$

4.2.3. *Proof of Theorem 4.6.* Let (M, g) satisfy the conditions of Theorem 4.6, then (M, g) also satisfies the hypothesis of Theorem 4.1. Therefore (M, g) splits as a warped product with metric of the form (4.1). Writing points of (M, g) as $(t, x) \in (-\pi/2, \pi/2) \times N$ it is not hard to see that the cosmological time function τ is given by $\tau(x, t) = t + \pi/2$. Define a new metric conformal $\tilde{g}$ to g by

$$\tilde{g} := \frac{1}{\cos(t)^2} g = -\frac{dt^2}{\cos(t)^2} + g_N = -(ds)^2 + g_N$$

where

$$s = \int_0^t \frac{dz}{\cos(z)} = \ln(\sec(t) + \tan(t)) = \ln(\sec(\tau - \pi/1) + \tan(\tau + \pi/2)).$$

Therefore $\tilde{g} = -ds^2 + g_N$ is the product metric on $(-\infty, \infty) \times N$. Then using the transformation rule (4.2) for $\|W_{\tilde{g}}\|_{\tilde{g}}^2$ and using that near $t = -\pi/2$, $\cos(t) = \cos(\tau + \pi/2) = \tau + O(\tau^3)$ so that

$$\lim_{s \to -\infty} \|W_{\tilde{g}}\|_{\tilde{g}}^2 = \lim_{t \searrow \pi/2} \cos(t)^4 \|W_g\|_g^2 = \lim_{\tau \searrow 0} \tau^4 \|W_g\|_g^2 = 0.$$

But as $\tilde{g}$ is the product metric the function $\|W_{\tilde{g}}\|_{\tilde{g}}^2$ will be constant along any of the lines $s \mapsto (s, x)$ in $(M, \tilde{g})$. Therefore the limit above implies $\|W_{\tilde{g}}\|_{\tilde{g}}^2 \equiv 0$ on M. As (M, g) is a Lorentzian manifold this in not enough to conclude $W_{\overline{g}} \equiv 0$. To get any conclusion at all we have to use that $(M, \overline{g})$ is a product metric.

Lemma 4.16. *Let (N, g_N) be a Riemannian manifold of dimension at least three, set $M := \mathbf{R} \times N$, and give M the Lorentzian product metric $g := -dt^2 + g_N$. Let R_{ABCD} be the curvature tensor of (M, g) as a $(0,4)$ tensor,*

R_{AB} the Ricci tensor and S the scalar curvature of (M,g). Let a and b be real numbers and set

$$V_{ABCD} := R_{ABCD} + a(g_{AC}R_{BD} + g_{BD}R_{AC} - g_{BC}R_{AD} - g_{AD}R_{BC}) \\ + b(g_{AC}g_{BD} - g_{AD}g_{BC}).$$

Then $\|V\|_g^2 \geq 0$ and equality on all of (M,g) implies (N,g_N) has constant sectional curvature. In particular the Weyl conformal tensor W of (M,g) is of this form so if $\|W\|_g^2 \equiv 0$ then (N,g_N) has constant sectional curvature.

Proof. This is a pointwise result so let $p \in M$ and choose an orthonormal basis $e_1,\dots,e_n$ of $T(M)_p$ so that $e_n = \partial/\partial t$. Then the vectors $e_1,\dots,e_{n-1}$ are tangent to the factor N. In what follows we will use the following range of indices

$$1 \leq A,B,C,D,A',B',C',D \leq n, \quad 1 \leq i,j,k,l,i',j',k',l' \leq n-1.$$

In the basis $e_1,\dots,e_n$

$$g_{AB} = g^{AB}, \quad g_{ij} = \delta_{ij}, \quad g_{in} = 0, \quad g_{nn} = -1.$$

With this notation we will show

$$\|V\|_g^2 = \sum_{i,j,k,l} (V_{ijkl})^2 + 4\sum_{i,j}(aR_{ij} + bSg_{ij})^2 \geq 0. \tag{4.9}$$

To do this split the sum for $\|V\|_g^2$ into two parts, the first only summing over the indices i,j,k,l,i',j',k',l' and the second where at least one index in the sum is equal to n.

$$\begin{aligned} \|V\|_g^2 &= \sum_{\substack{A,B,C,D \\ A',B',C',D'}} g^{AA'}g^{BB'}g^{CC'}g^{DD'}V_{ABCD}V_{A'B'C'D'} \\ &= \sum_{\substack{i,j,k,l \\ i',j',k',l'}} g^{ii'}g^{jj'}g^{kk'}g^{ll'}V_{ijkl}V_{i'j'k'l'} + \sum\nolimits_2 \\ &= \sum_{i,j,k,l} (V_{ijkl})^2 + \sum\nolimits_2 \end{aligned} \tag{4.10}$$

We now consider the terms that occur in $\sum_2$. Because the metric $g = -dt^2 + g_N$ is a product metric and n corresponds to the direction of $e_n = \partial/\partial t$ we have

$$R_{ABCn} = 0, \quad R_{An} = 0.$$

All the terms in the sum $\sum_2$ have at least one index equal to n. By the symmetries of the curvature tensor we can consider the case where $D = n$. Using that (M,g) is a product $R_{ABCn} = 0, R_{An} = 0$ so

$$V_{ABCn} = a(g_{Bn}R_{AC} - g_{An}R_{BC}) + bS(g_{AC}g_{Bn} - g_{An}g_{BC}).$$

From this we see $V_{ABnn} = 0$ and likewise $V_{nnCD} = 0$. Also $V_{ijkn} = 0$ Thus if a term in $\sum_2$ is nonzero, exactly two of A,B,C,D are equal to n and

moreover the case $A = B = n$ and $C = D = n$ give zero terms. Consider the case $B = D = n$. Then

$$V_{inkn} = -aR_{ik} - bSg_{ik}.$$

The sum over the terms in $\sum_2$ where $B = D = n$ is then

$$\sum_{\substack{i,j \\ A',B',C',D'}} g^{iA'} g^{nB'} g^{kC'} g^{nD'} V_{ABCD} V_{A'B'C'D'} = \sum_{i,j} (g^{nn})^2 V_{inkn} V_{inkn}$$

$$= \sum_{i,k} (aR_{ij} + bSg_{ik})^2.$$

The calculations for the three cases $A = D = n$, $A = C = n$, and $B = C = n$ give the same result. As these are the only cases leading to nonzero terms in the sum $\sum_2$ we have $\sum_2 = 4\sum_{i,j}(aR_{ij} + bSg_{ij})^2$. Using this in (4.10) completes the verification of (4.9).

Now assume that $\|V\|_g^2 = 0$ at a point. Then equation (4.9) implies that $aR_{ij} + bSg_{ij} = 0$. Therefore

$$a(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{jk}R_{il} - g_{il}R_{jk}) = -2b(g_{ik}g_{jl} - g_{il}g_{jk})$$

But (4.9) also implies $V_{ijkl} = 0$. Using these facts in the definition of V to solve for R_{ijkl} gives

$$R_{ijkl} = bS(g_{ik}g_{jl} - g_{il}g_{jk})$$

for $1 \le i, j, k, l \le n - 1$. But as $(M, g) = (\mathbf{R} \times N, -dt^2 + g_N)$ is a product of with (N, g_N) the curvature tensor R of (M, g) and the curvature tensor R^N of (N, g_N) are related by $R^N_{ijkl} = R_{ijkl} = bS(g_{ik}g_{jl} - g_{il}g_{jk})$. Therefore if $\|V\|_g^2 \equiv 0$ then the curvature tensor of (N, g_N) is of the form $R^N_{ijkl} = bS(g_{ik}g_{jl} - g_{il}g_{jk})$. As the dimension of N is ≥ 3 a well known theorem of Schur (cf. [22, Thm 2.2 p. 202]) implies (N, g_N) has constant sectional curvature. This completes the proof. $\square$

We now complete the proof of Theorem 4.6. We have shown that under the hypothesis of the theorem the metric $\tilde{g} = -ds^2 + g_N$ has $\|W_{\tilde{g}}\|_{\tilde{g}}^2 \equiv 0$. Then the lemma implies that (N, g_N) has constant sectional curvature. The completes the proof.

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